Higher order beliefs and their effect on game analysis

Thesis

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Section 1

Introduction

Is discerning the outcomes of the behavior of market players through models of incomplete information undependable as a means of forecasting economic behavior? In microeconomics, games with incomplete information are used regularly, however many models constructed in this manner reach a level of complexity that can make interpreting results difficult. Inexplicable anomalies of the model—so-called sunspots—are an indication that models of incomplete information are in and of themselves unfit to base presumptions about future outcomes. The central argument of this thesis is that models that observe the effect of higher order beliefs can be used as a means of predicting certain phenomena accurately, and provides an overview detailing the method of its use. This paper is divided into four sections. In the second, I present an introduction to the basic concepts of game theory, with emphasis on the methods, and notations pertaining to complete and incomplete information strategic games. With normal form finite strategic games as its starting point, it goes on to describe the method of calculating Nash equilibria, then touches on the subject of models of interaction with incomplete information, finally closing with the subject of knowledge and the types of knowledge that players can possess. The second section also outlines the current scientific stance towards the aforementioned. The indeterminacy of outcomes is the result of two prior assumptions:

1. knowledge fundamental to the game’s nature is common knowledge, and
2. players know each other’s strategies in equilibrium.
If these assumptions are reviewed, it must be noted that certain outcomes become easier to predict if the probable set of information the player possesses narrows his scope of choices. Because all players have only incomplete information to base their actions on, it is reasonable to assume that some of the models observable have risk-dominant equilibria as opposed to payoff-dominant. The phenomenon of risk dominant actions in game theory are demonstrated in Global game models, the discussion of which constitutes section 3. Risk-dominance is an important factor because in models where players make choices under the influence of noise, they might make decisions in terms of what information they actually have possession of; for example choosing to ignore strategies where expected payoff is higher but still unknown, as opposed to a specifically forecasted payoff. Games where player decisions are influenced to the point of contradicting traditional methods of discerning outcomes because of the phenomena of higher order beliefs do so on this basis; if a player does not possess information pertaining to the information another player possesses, he will modify his dominant strategy accordingly. Section 4 is a summary of the effect higher order beliefs have on player decision in general which aims to highlight theoretical and real world examples, and the relevance of the theory’s use.
Section 2

The Basics

2.1 Strategic Games

The basic model of strategic interaction is known as a strategic game (von Neumann and Morgenstern 1928), otherwise known as a game in normal form. The concept of this model is as follows: given a certain state of affairs, each player simultaneously makes a decision as to which action he will choose, taking into consideration the respective payoff attributed to it, as well as their own preferences.

**Definition 2.1.1** The constituents of a strategic game are

- a finite $N$ number of players
- a set of $A_i$ actions available to these players, where $i \in N$
- player $i$’s $\succ_i$ preference relation on $A = \times_{j \in N} A_j$

The rational decision-maker’s preferences are specified via the use of a utility function.

**Definition 2.1.2** A utility function quantifies the preference of the players by binding a value $x, y$ to all $C$ consequences ($U : C \to \mathbb{R}$) such that $x \succ y$ if and only if $U(x) \geq U(y)$.

If the observed game has a set of consequences which depend on exogenous circumstances, these can be incorporated by introducing the space $\Omega$; this redefines the solution function to $g : A \times \Omega \to C$. The consequence function is that of $g(a, \omega)$, the action profile is $a \in A$ and the realization of the random variable is $\omega \in \Omega$. Thus
it follows, that a profile of actions induces a lottery on $C$, and for each player, the preference relation must be specified on these lotteries.

**Definition 2.1.3** A lottery is a randomized event pertaining to the outcome of one’s decision. By pairing each possible outcome with a probability, and then calculating the linear combination of an action profile ($L = \sum p_i A_i$), it becomes possible to compare action profiles, and thereby determine player strategies (von Neumann and Morgenstern 1947).

Preference relations can also be substituted with a Payoff function $u_i : A \to \mathbb{R}$ if they conform to the Von Neumann-Morgenstern principles of completeness, transitivity, continuity and independence. In Figure 2.1, player one’s actions are the rows ($T, B$), player two’s are the columns ($L, R$), and $a_{r,c}$ are the corresponding payoffs to their decision.

### 2.2 Nash Equilibrium

The concept of Nash equilibrium is the most commonly used solution when analyzing strategic games. The core notion is finding a state of play where each the player acts rationally while perfectly informed of the other players’ motivation and actions. His decisions will be aimed at achieving the highest payoff possible. The action with which he can achieve this payoff is called the player’s best response (Nash 1950).

**Definition 2.2.1** A Nash Equilibrium of a strategic game $\langle N, (A_i), (\succ_i) \rangle$ is a profile $a^* \in A$ of actions with the property that for every player $i \in N$

$$ (a^*_i, a^*_i) \succeq (a^*_{i-1}, a_i) \text{ for all } a_i \in A_i \text{ and all } i \in N $$

$(\text{def. 14.1 – (Osborne and Rubinstein 1994)})$
Definition 2.2.2 The best response function \( B_i \) is the set of player \( i \)'s best actions given any \( a_{-i} \), and the following equivalence holds true:

\[
B_i(a_{-i}) = \{ a_i \in A_i : (a_{-i}, a_i) \succeq (a_{-i}, a_i') \text{ for all } a_i' \in A_i \}.
\]

Then, a Nash equilibrium is a profile \( a^* \) of actions for which

\[
a_i^* \in B_i(a_{-i}^*), \text{ for all } i \in N.
\]

(equations 15.1-15.2 – (Osborne and Rubinstein 1994))

The state or states of play which would maximize each player' payoff is called the steady state of play. It is assumed that straying from the optimum decision is in no way profitable for either player. \( B_i \) is player \( i \)'s best response function. To find a steady state, the analyst calculates the best response function for each player, then determines a profile of actions \( a^* \) containing the optimum decisions \( a_i^* \) which all must be from the set \( B_i(a_{-i}^*) \). Osborne & Rubinstein cite several examples which can be analyzed in the above described manner. To summarize:

1. **Bach or Stravinsky? (BoS, Bush vs. Saddam, Battle of the Sexes)** –
   Because players have conflicting interest, they cannot agree to either choice; however, because making the same decision definitely has a higher yield for both players, one of them is forced to compromise. Thus, this game has two steady states, \((B, B)\) and \((S, S)\).

   ![Table 2.2: Bach or Stravinsky? (BoS) - example 15.3](image)

<table>
<thead>
<tr>
<th></th>
<th>Bach</th>
<th>Stravinsky</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bach</td>
<td>2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>Stravinsky</td>
<td>0, 0</td>
<td>2, 1</td>
</tr>
</tbody>
</table>

   Figure 2.2: Bach or Stravinsky? (BoS) - example 15.3

2. **A coordination game** – Like the previous game, it is in the players’ interest to make the same decision in a coordinated fashion. This time, both players favor the same decision, so this game has one steady state, \((C_1, C_1)\).

3. **The Prisoner’s Dilemma** – It is in the players’ interest to make the same decision, and both prefer the same choice. However, the incentive to deviate from a coordinated effort forces all players to prefer the choice with lower payoff. The sole steady state is \((C, C)\).
4. **Hawk - Dove** – In this game, it is in the player’s best interest to choose opposing strategies - that is to be hawkish, if the other player is dovish, and vice-versa. This game also has two steady states, \((H, D)\) and \((D, H)\).

![Hawk-Dove Game](image)

5. **Matching Pennies** – A coin is flipped twice, and player one wins a dollar from player two if it fall on the same side both times, while player two wins a dollar from player one if it falls on different sides. This is what is known as a *strictly competitive*, or *zerosum* game. It has no Nash equilibrium.

![Matching Pennies Game](image)

As demonstrated in the Matching Pennies game, not all feasible models necessarily have a Nash equilibrium. Proof of its existence is garnered by showing that a profile \(a^*\) of actions exist, such that \(a_i^* \in B_i(a_{-i}^*)\) for all \(i \in N\) (Osborne and Rubinstein 1994). This is done by defining the set-valued function \(B : A \rightarrow A\) by \(B(a) = \times_{i \in N} B_i(a_{-i})\), and converting the profile \(a^*\) as the vector \(a^* \in B(a^*)\). A *fixed point theorem* is then used to prove the existence of \(a^*\). Osborne and Rubinstein use Kakutani’s fixed point theorem (Kakutani 1941), the matter of which is not discussed here.
2.3 Bayesian games

Analyses of models based on real life demand that the analyst factor in the players limited knowledge of exogenous conditions, which may affect the outcome of their chosen action. A Bayesian game is a model of behavior based on the finite strategic games model that the analyst can use to calculate the effect of "unknowns" in the system (Harsányi 1967/68).

**Definition 2.3.1** A Bayesian game requires a finite set

- of **players** $N$
- of **states** $\Omega$.

Each player $i \in N$ has

- a set of **actions** $A_i$
- a set of available **states of nature** $\Omega$
- a finite set of **observable signals** $T_i$, and the corresponding **signal function**
  $\tau_i : \Omega \rightarrow T_i$
- the prior belief of players of the probabilities $P_i$ on $\Omega$, for which
  $p_i(\tau_i^{-1}(t_i)) > 0$ for all $t_i \in T_i$
- a preference relation $\succsim_i$ of the players over $A \times \Omega$, where $A = \times_{j \in N} A_j$.
  (definition 25.1 - (Osborne and Rubinstein 1994))

To model the play of the game, one chooses a state $\omega \in \Omega$ which is realized by probability $p_i$, and then observing the signal function player $i$ observes before making a decision. The observable signals, or **types**, are then deduced by selecting those which player $i$ can choose from $\tau_i$. Assuming that all players assign positive
prior probabilities $p_i$ to all types, a set $\tau_i^{-1}(t_i)$ is selected as the current state of nature, also known as player $i$’s posterior belief, to which said player assigns the probability $p_i(\omega)/p_i(\tau_i^{-1}(t_i))$ assuming $\omega \in \tau_i^{-1}(t_i)$, and 0 if otherwise. Due to the players’ limited information about the state of nature, a profile of preference relation must be included over lotteries of $A \times \Omega$, the set of expected outcomes for each action. To calculate the action player $i$ makes, the analyst has to find the model’s Nash equilibrium.

**Definition 2.3.2** A Bayesian Nash equilibrium of a Bayesian game is a Nash equilibrium of a strategic game with the following attributes:

- The set of players is the set of all pairs $i, t_i$ for $i \in N$ and $t_i \in T_i$
- The set of actions for each player is $A_i$
- The preference ordering $\succsim_{i,t_i}$ of each player is defined by

$$a^* \succsim_{i,t_i} b^* \text{ if and only if } L_i(a^*, t_i) \succsim_i L_i(b^*, t_i),$$

where $L_i(a^*, t_i)$ is the lottery over $A \times \Omega$ that assigns probability $p_i(\omega)/p_i(\tau_i^{-1}(t_i))$ to $((a^*(j, \tau_j(\omega)))_{j \in N}, \omega)$ if $\omega \in \tau_i^{-1}(t_i)$, and 0 otherwise. (Def. 26.1 - (Osborne and Rubinstein 1994))

This model template can be used not only to determine equilibrium in models where players have incomplete information about each other’s payoffs, but when they are unsure of each other’s knowledge as well. To do this, it is necessary to express the players’ uncertainty about their payoffs by introducing $\theta \in \Theta$. Then, let player $i$’s beliefs be a probability distribution over $\Theta \times X_i$, where $X_i$ are a set of possible beliefs. Thus, the state space used for analysis will be $\Omega = \Theta \times (\times_{i \in N} X_i)$.

Each player will choose a strategy profile $s_i(\theta_i)$ that gives the players’ strategy choice for each realization of his type $\theta_i$. Player $i$’s strategy profile is then given by

$$U_i(s_1(a), \ldots, s_i(a)) = E_\theta[U_i(s_1(\theta_1), \ldots, s_N(\theta_N), \theta_i)].$$

Let $\Sigma$ be the set of feasible strategies available to each player. If the preference ordering of each player is exchanged for a utility function as defined in Def. 2.1.2, this definition can be rephrased in the following form:
Definition 2.3.3 A Bayesian Nash equilibrium of a Bayesian game \( \langle N, (S_i), U_i(a), \Theta \rangle \) is a strategy profile \( s_1(a), \ldots, s_i(a) \) that constitutes a Nash equilibrium of the game \( \langle N, (\Sigma_i), (U_i(a)), T_i \rangle \) like so:

\[
U_i(s_i(a)) \geq U_i(s'_i(a))
\]

for all \( s'_i \in \Sigma_i \), where \( U_i(a) \) is defined as written above (def. 8.E.3 - (Mas-Colell, Whinston and Green 1995)).

To determine the Nash equilibrium of any strategic game requires that it be modeled in normal form. The normal form of a Bayesian game can be calculated via interpreting its extensive form (Kuhn 2003). To represent games in extensive form, it is necessary to introduce the concept of a game tree.

Definition 2.3.4 A game tree \( K \) is a directed graph whose nodes are positions in a game, and whose edges are moves.

![Figure 2.7: A basic game tree](image)

A complete game tree is a map-like representation of all possible choices open to both players (von Neumann and Morgenstern 1928). Turns of play are represented by the amount of levels or plies the tree has. The vertices of a game tree are called positions, the actions taken are called plays, and the ones that are not are moves, which are denoted by the generic symbol \( W \). The edges originating in any given \( W \) are alternatives at \( W \). An extensive game carries the following specifications, besides being in the form discussed so far:
Definition 2.3.5 An n-person **game in extensive form** is a game tree with the following attributes:

1. a finite set of nodes $\chi$, a finite set of possible actions $A$, and a finite set of players $N$

2. a function $p : \chi \rightarrow (\chi \cup \emptyset)$ specifying a single immediate predecessor of each node $x$, which is non-empty except in the case of the initial node $x_0$

3. a function $\alpha : \chi(x_0) \rightarrow A$ giving the action that leads to any non-initial node $x$ from its immediate predecessor $p(x)$

4. a collection of information sets $\eta$, and the function $H : \chi \rightarrow \eta$ assigning each decision node $x$ to an information set $H(x) \in \eta$

5. a function $\iota : \eta \rightarrow (0, 1, \ldots, N)$ assigning each information set in $\eta$ to player $i$

6. a function $\rho : \eta_0 \times A \rightarrow (0, 1)$ assigning probabilities to actions at information sets where nature moves

7. a collection of payoff functions $U = (U_1(a), \ldots, U_N(a))$ assigning utilities to the players for each terminal node that can be reached, $U_i : T \rightarrow \mathbb{R}$

Formally, a game in extensive form has the following building blocks:

$$\langle \chi, A, N, p(\cdot), \alpha(\cdot), \eta, H(\cdot), \iota(\cdot), \rho(\cdot), U \rangle.$$ 

(Mas-Colell et al. 1995)

The progression of the game is as follows: player one chooses an alternative, and player two reacts with a choice of his own. Should move $X$ be a chance move, an alternative is chosen by a chance device assigned to it. Thus, all player decisions (or probable states) can be followed through the path they take. The players gain the payoff corresponding to the last vertex of this path. For example, the Matching Pennies game from the previous section can be represented like so:

Definition 2.3.3 suggests that each type of player $i$ can be interpreted as a separate player who maximizes his payoff given his conditional probability distribution over the strategy choices of his rivals.
With this tool in the analyst’s arsenal, he can find Bayesian games’ normal form, which is essential to determining its equilibrium. Consider the following modification of the BoS game, borrowed from a lecture by Prof. Dr. Dezső Szalay of the University of Bonn.[1] In this variant, player one’s preferences are common knowledge, however, player two’s are only known to herself. The matter of uncertainty stems from player two’s preference; namely, if she prefers to go out with player one or does not. Thus, there are two \( \omega \) and these can be represented with two normal form, two-player games.

\[
\begin{array}{c|cc}
 & B & S \\
\hline
B & 2, 1 & 0, 0 \\
S & 0, 0 & 1, 2 \\
\end{array}
\]

Figure 2.9: modified BoS: state \( \alpha \)

\[
\begin{array}{c|cc}
 & B & S \\
\hline
B & 2, 0 & 0, 2 \\
S & 0, 1 & 1, 0 \\
\end{array}
\]

Figure 2.10: modified BoS: state \( \beta \)

Player one doesn’t know whether the state of nature is \( \alpha \) or \( \beta \). To factor in the uncertainty stemming from this, proper representation necessitates the introduction of a new player, Nature, which will play either state with a positive prior probability.

For convenience, let this prior probability be $\frac{1}{7}$. This game is represented in extensive form in Figure 2.11.

Figure 2.11: modified BoS in extensive form

To find the equilibrium of the game, one has to find its normal form. The payoffs are gained by averaging the linear combination of the four strategies available to player two times the two available to player one.

<table>
<thead>
<tr>
<th></th>
<th>$B_1$, $B_2$</th>
<th>$B_1$, $S_2$</th>
<th>$S_1$, $B_2$</th>
<th>$S_1$, $S_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$</td>
<td>2, 0.5</td>
<td>1, 1.5</td>
<td>1, 0</td>
<td>0, 1</td>
</tr>
<tr>
<td>$S$</td>
<td>0, 0.5</td>
<td>0.5, 0</td>
<td>0.5, 1.5</td>
<td>1, 1</td>
</tr>
</tbody>
</table>

Figure 2.12: modified BoS in normal form

Once we have the normal form game, calculating the steady states is no different from any other finite strategic game. Player two’s best response

- to $B$ is $B_1$, $S_2$,
- to $S$ is $S_1$, $B_2$,

while Player one’s is

- $B$ to $B_1$, $S_2$,
- and likewise for $S_1$, $B_2$,

thus, the game has only one steady state where player one plays $B$ and player two plays $B_1$, $S_2$. 

12
2.4 Common Knowledge

There was mention earlier of player one’s preferences being "common knowledge" in the previous example. To understand the concept of higher order beliefs, it is first necessary to understand what this means from a game theoretic viewpoint. Foremost, it is important to draw a distinction between mutual (also called "almost common") and common knowledge, for the former is merely the fact that a set of information \( I \) is known by all players. Apart from being mutual, common knowledge is information which is known by all individuals within the game to be known by all other individuals, is known by all individuals to be known by all individuals to be known by all individuals, and so on ad infinitum (Aumann 1976).

Definition 2.4.1 A set of information \( I \subseteq \Omega \) is common knowledge between players 1 and 2 in the state \( \omega \in \Omega \) if \( \omega \) is a member of every set in the infinite sequence \( K_1(I), K_2(I), K_1(K_2(I)), K_2(K_1(I)), \ldots \), where \( K_1, K_2 \) is players 1 and 2’s knowledge functions. (definition 73.1 – (Osborne and Rubinstein 1994))

An alternate definition uses the information functions of the players:

Definition 2.4.2 Let \( P_i \) be the information function of player 1. An event \( F \subseteq \Omega \) is self-evident between one and two if for all \( \omega \in F \) we have \( P_i(\omega) \subseteq F \) for all \( i = 1, 2 \). An event \( E \subseteq \Omega \) is common knowledge between one and two in the state \( \omega \in \Omega \) if there is a self-evident event \( F \) for which \( \omega \in F \subseteq E \). (definition 73.2 – (Osborne and Rubinstein 1994))

The fundamental difference between mutual and common knowledge can completely change the steady state of a game; this is where this paper’s main topic, higher order beliefs come into play. Beliefs of higher order are characterized by the singular members of the infinite sequence of information in the previous definition. What players know or believe about each other’s rationality, actions, knowledge, and beliefs is called the players’ epistemic state (Aumann and Brandenburger 1995). While the thought that this subtle difference between types of knowledge might sound counter-intuitive, it can befuddle completely the analysis of an otherwise highly representative game model.
Section 3

Global Games

3.1 Risk dominance vs. Payoff dominance

A variation on the generic coordination game described in section 2.2 is the Stag hunt model (Skyrms 2004). In this game, it is assumed that player decisions are not common knowledge. Players have incentive to hunt rabbit, because they have a larger pool of prey that is easier to kill, and have no competition if the other player chooses to hunt stag. Therefore, this game has only one steady state - both players hunt rabbit. This game is an analogy to the Prisoner’s Dilemma, and both are risk dominant because of the absence of credible commitment on the players’ part.

\[
\begin{array}{c|cc}
& Stag & Rabbit \\
Stag & 5, 5 & 1, 4 \\
Rabbit & 4, 1 & 3, 3 \\
\end{array}
\]

Figure 3.1: the Stag hunt

Generally, coordination problems can be represented with a payoff matrix such as the one in figure 3.2, where capital letters stand for player one’s payoffs, and the following inequalities hold: \( A > B, D > C \). The same is true for player two’s payoffs \( a, b, c, d \).

Thus, the only pure Nash equilibria are \((S, S)\) and \((R, R)\). However in an incomplete information setting, the strategy \((R, R)\) is risk dominant, as defined below:

**Definition 3.1.1** In a strategic game, \((R, R)\) is risk dominant over \((S, S)\) if the product of the losses stemming from the deviation of players from acting in a
coordinated fashion is higher for \((R, R)\), in other words, if the inequality

\[(C - D)(c - d) \geq (B - A)(b - a)\]

is true, \((S, S)\) is **payoff dominant** over \((R, R)\) if \(A \geq D, a \geq d\) and at least one of these two inequalities are strict, i.e. \(A > D\) or \(a > d\) (Harsányi and Selten 1988).

A risk dominant game will, by virtue of the players opting for smaller than maximum payoffs, alter the steady state of the coordination game. As such, using it in its present form may lead to false or uninterpretable results. To better illustrate this phenomenon, the following section touches on a type of game where player’s react to possibly correlated signals of the underlying state of nature.

### 3.2 The Global Games Model

The idea of Global Games was first discussed by Hans Carlsson and Eric van Damme, whose underlying thought was to find optimal strategic behavior of players in the economy while observing all possible infinite hierarchies of beliefs. Although this is intractable given the high complexity and infinite permutations of possibilities, it can be approximated by using an incomplete information model that is rich enough to capture the environment that decisions are made in. The central concept of the game is that each player makes his decision while uncertain of the state of nature, and observes a signal of a state \(\theta\) with a small amount of noise, where it is assumed that all players possess the same noise technology, i.e. the probability that their knowledge is correct or incorrect is the same as all other players’. In this setting, players choose the action that is the best response to a uniform belief over the proportion of his opponents choosing their own respective strategies. As such, players who face uncertainty concerning the underlying state of nature will believe that the aforementioned proportion is uniformly distributed over the unit interval, and will
choose their actions accordingly.

If the actions of other's in an incomplete information game can be deduced or approximated, then so can its outcomes, and this paves the way for systematic analysis of economic questions which are otherwise unfeasible. Consider the following example (Carlsson and van Damme 1993a): Two players are deciding to invest. There is a risky action (invest), and a safe action (not invest); the risky alternative yields a higher payoff, which is a conventional expectation as far as investments go. The payoff matrix is

\[
\begin{array}{c|cc}
\text{Invest} & \text{Invest} & \text{Not Invest} \\
\hline
\text{Invest} & \theta, \theta & \theta - 1, 0 \\
\text{Not Invest} & 0, \theta - 1 & 0, 0
\end{array}
\]

Figure 3.3: A symmetric binary action global game

Supposing that \( \theta \) is common knowledge, there are three possible solutions to this game:

1. All players will invest if \( \theta > 1 \)
2. There are two steady states \((I, I)\) and \((NI, NI)\), if \( \theta \in [0, 1] \)
3. Neither player will invest if \( \theta < 0 \)

However, the players cannot be certain about \( \theta \). Thus, they will turn to their respective sources of information, and observe a private signal. For player \( i \), this private signal is \( x_i = \theta + \epsilon_i \) where \( \epsilon_i \) is the amount of noise player \( i \) will experience. Each \( \epsilon \) is independently normally distributed with mean 0 and standard deviation \( \sigma \). Thus, a player observing a signal \( x \) is assumed to consider \( \theta \) to be normally distributed with mean \( x \) and standard deviation \( \sigma \), and it is also assumed that he will therefore expect the other player’s signal \( x' \) to be normally distributed with mean \( x \) and standard deviation \( \sqrt{2}\sigma \).

**Definition 3.2.1** A strategy profile is a function which specifies an action \( a_i \) for each possible private signal the player may observe. This involves determining a cutoff point \( k \) which when observed will change the action player \( i \) will choose in hope of attaining the maximum possible payoff.
In this game’s case, the following players choose based on the following strategy profile:

\[ s(x) = \begin{cases} 
  I, & \text{if } x > k \\
  NI, & \text{if } x \leq k 
\end{cases} \]

This strategy profile is referred to as the *switching strategy around k*. A rational player will assume that his competitors will also be observing a switching strategy, and will assign the probability \( \Phi \left( \frac{k-x}{\sqrt{2}\sigma} \right) \) (where \( \Phi(\cdot) \) is a cumulative distribution function of the standard normal distribution for them observing a signal \( x < k \). Thus, if \( \sigma = 0 \) (\( k = x \)), the probability of the other player investing is exactly \( \frac{1}{2} \).

Let \( b(k) \) be the unique value of \( x \) which solves \( x - \Phi \left( \frac{k-x}{\sqrt{2}\sigma} \right) = 0 \). This is a unique strategy which survives iterated deletion of strictly interim dominated strategies, because when plotting \( b(k) \), the left side is strictly increasing in \( x \), while strictly decreasing in \( k \) on the right side.

![Figure 3.4: The cumulative distribution function (c.d.f) over k](image)

This in turn implies that \( b(\cdot) \) is strictly increasing, so the player’s best response is to follow a switching strategy with cutoff point \( b(k) \), is his opponent observes the same with cutoff point \( k \). Carlsson and van Damme present the solution, that the following strategy profile \( s \) – having survived \( n \) rounds of iterated deletion – holds true for expected player behavior:

\[ s(x) = \begin{cases} 
  I, & \text{if } x > b^{n-1}(1) \\
  NI, & \text{if } x \leq b^{n-1}(0) 
\end{cases} \]

where given \( n = 1 \), the expected value of \( \theta \) is less than zero (making NI the dominant
strategy), and both $b^{n-1}(1)$ and $b^{n-1}(2)$ tend to $\frac{1}{2}$ as $n$ tends to $\infty$ (Carlsson and van Damme 1993a). There is a unique equilibrium in this game whereby players will only invest should they observe a signal $k$ larger than $\frac{1}{2}$, because investing is a risk dominant action when $\theta \geq \frac{1}{2}$ (and not investing is risk dominant if $\theta \leq \frac{1}{2}$). Thus, players’ decisions will always be influenced by the possibility that their counterparts will have a dominant strategy to choose either action, without regard for the size of $\sigma$. With this information, the c.d.f over $\theta$ can be written like so:

$$\Phi\left(\frac{\frac{1}{2} - \theta}{\sigma}\right)$$

So, as stated above, even if the standard deviation of $\theta$ is negligible, all players will base their decisions on what they would expect of each others decisions. If player one thought player two would play NI while observing $x < b^{n-1}(1)$, he would be compelled to also play NI given his own observed $x$ if it was less than $b(b^{n-1}(1))$. In other words, players choose actions based on higher order beliefs concerning their competitors.
Section 4

Higher Order Beliefs – an Overview

4.1 Theoretical Example

Consider a coordination game where two generals are planning to attack a common enemy from different locations. Neither general knows of the enemy’s strength – thus, there are two $\omega$, both with a prior probability of $\frac{1}{2}$. It is known, that, regardless of the state of nature, neither army is strong enough to defeat the enemy acting alone, so coordination is their only option. The Coordinated Attack Problem (Halpern 1986) is analogous to the Electronic Mail Game (Rubinstein 1989).

\[
\begin{array}{c|cc}
\text{Attack} & \text{Not Attack} \\
\hline
\text{Attack} & -2, -2 & -2, 0 \\
\text{Not Attack} & 0, -2 & 0, 0 \\
\end{array}
\]

Figure 4.1: The Coordinated Attack Problem – state $\alpha$, where the enemy is strong

\[
\begin{array}{c|cc}
\text{Attack} & \text{Not Attack} \\
\hline
\text{Attack} & 1, 1 & -2, 0 \\
\text{Not Attack} & 0, -2 & 0, 0 \\
\end{array}
\]

Figure 4.2: The Coordinated Attack Problem – state $\beta$, where the enemy is weak

Player one will only consider attacking if he receives a reliable signal that the state of nature is $\beta$. Even so, this information is not common knowledge (player two might not know, or he might not know that player one knows). Should the state of nature be $\beta$, and player one is informed of this, then for sake of a coordinated attack,
he sends a messenger to player two to ascertain his cooperation. However, there is a small, positive $\epsilon$ probability that the messenger is captured while in transit; therefore, player one also needs to ascertain that player two has indeed been delivered the message, and he tasks his messenger with bringing back confirmation. Thinking along the same lines, player two also wishes to have confirmation that player one has received confirmation that player two knows the state of nature is $\beta$. In this fashion, the players send a $T$ number of messages until the messenger is captured, or – after an infinite number of iterations – the contents of that message become common knowledge. Both players will choose an action only if communication between them has ceased.

Now let $S_1, S_2$ be a Nash equilibrium such that $S_1(0) = NA$. Any player waiting to receive confirmation of their sent message will attach a probability $p$ of the other player sending that confirmation, and $p\epsilon$ that the reply was lost. Because both players are committed to an attack only once their plan of coordination and the state of nature become common knowledge, they will play NA for all $T_i < t$, where $t$ is the last message that any player has sent. Before receiving the reply of his opponent, a player cannot be certain if his latest message arrived at the other general's camp, or if it was the other player’s message that was intercepted (i.e $T_t = t$, or $T_t = t - 1$, respectively.) Therefore, the players each attach a conditional probability $z = \epsilon/[(1 - \epsilon)\epsilon] > \frac{1}{2}$ after his $t$-th message that $T_t = t - 1$, which means they assume the chances of their last message not arriving is higher than the chances of their opponent’s message not arriving. They will therefore be confident that – having not received confirmation – their opponent will play NA. Then, if the player

- plays A, his expected payoff is $z(-2) + (1 - z)(1)$, and if he
- plays NA, his utility is 0.

Having established that $z > \frac{1}{2}$, the players’ only rational decision is to play $S_1(t) = A, S_2(t) = A$. This example illustrates the effect players’ epistemic state has on equilibria; no matter how small $\epsilon$ is, the dominant strategy will always be the one with a lower payoff in state $\beta$, whereas the phenomenon doesn’t affect the equilibrium in state $\alpha$. 

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The Coordinated attack problem can be rewritten in a generic form like so:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>M, M</td>
<td>-L, 0</td>
</tr>
<tr>
<td>NA</td>
<td>0, -L</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

Figure 4.3: The C.A.P - state $\alpha$

where $L > M$. However, a different result arises if we assume that both players’ payoffs are $-L$ in the event of an uncoordinated attack. An equilibrium can be found wherein player two will play $A$ once he gets at least one message, provided that there is a small enough $\epsilon; \epsilon < M/(M + L)$ to be exact (Rubinstein 1989).

4.2 Conclusion

Morris and Dasgupta¹ write of a few real world situations where higher order beliefs cause players to act differently in an incomplete information setting. A few of these are:

- some investors withdrew their capital from Brazil following the Asian crises of 1997 not because they thought the economic links between Brazil and Asia were disproportionately overestimated, but because they expected that others might do so,

- some lenders make excessive margin calls and seek to liquidate collateral when the hedge fund portfolio they invested in experiences a negative shock because they thought others might do so, and were motivated to get out as fast as possible,

- apparently irrelevant news about the economy leads some firms to reduce their investment and can have the economy spiral into full blown recession, because they feel others might think the news is relevant,

- some investors may pay highly inflated prices for certain Internet stocks, because they expect other investors might feel that the quantity of future dividends open up the possibility to make short run speculative profits, and will therefore justify these prices.

¹http://personal.lse.ac.uk/DASGUPT2/hob_main.html
As demonstrated, it is not common knowledge if the public signals observed by the players is payoff irrelevant or not. The "sunspot" explanation (which explains currency crises, bank runs, recession and other phenomena) is merely a placeholder of the true higher order beliefs explanation. The increasing speed of the flow of information in the 21st century means that the set of almost common knowledge a player possesses is continually growing as he moves forward in time, and this fact necessitates thorough analysis of the effect a player’s epistemic state has on forming their choices. Thus, it is the author’s view that the effects of beliefs of higher orders should be considered in all game models.
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