Modelling of insider information using enlargement of filtrations
Bennfentes információ modellezése filtráció bővítéssel

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Chapter I

Introduction

Most of the basic financial theories assume that the market satisfies strong efficient market hypothesis, which states that the prices of financial instruments reflect every information, even insider information is reflected in the prices. However, empirical studies show that this assumption is violated and there are insider information that is not incorporated in the asset prices. Therefore insider information may mean an advantage to its owner, but the most important question is how the insider could utilize it and how much additional expected utility could it assure.

To answer this question, at first let me summarize the existing models dealing with this topic, my focus being on the framework defined by Amendinger et al. (1998) as this would be the base of this thesis. They focus on two investors, a regular one and an insider: the latter knows some insider information that is not available to the regular investor. This difference can be covered mathematically by using the base filtration for the regular investor and defining an enlarged one for the insider. They assumed that the utility of both investors could be written as the logarithm of the terminal wealth. This is a very practical assumption as it simplifies the optimization problem for both market participants utilizing the exponential structure of the wealth process. Therefore the question: how much additional expected utility is coming for the insider from the additional information will be easy to answered.

For the additional utility to be a finite value, some additional conditions have to be met. However, before getting to that point a discounted wealth process is still needed and its components require more steps to be defined. The first problem is that instead of having only a Brownian filtration and a complete market, with the enlarged filtration the new setup is an incomplete market with a different kind of filtration. Therefore it is needed to introduce a martingale preserving measure under this enlargement. Mathematical results, mostly based on Jacod (1985) will enable us to do so, which will be introduced on the go. There is also an alternative way to calculate the additional logarithmic utility in cases where Jacod’s theorem not necessarily holds.

I will also examine ideas covered in Imkeller et al. (2001) and Imkeller (2003) and how they could improve this model. They introduce an alternative methodology
that can even treat a few of those cases, where Jacod’s condition is not necessarily applicable. I will show the basics of this approach and how to calculate the additional utility of an insider with enlargement of filtrations technique when Jacod’s condition is not applicable. I will introduce Clark-Ocone formula, which will be used in this framework. To understand the formula, it is also needed to get familiar with the basics of Malliavin calculus which will be also presented later on.

At the end, I will adapt these theoretical results to do my own calculation with different assumptions on the insider information. I will investigate the well-known Black-Scholes market and calculate the optimal strategy for both type of investors and their expected logarithmic utility. To do so, I will need to identify the appropriate density process for the change of measures, so that I could treat the Brownian motion of the basic filtration under the enlarged measure. My calculations will cover the cases of the additional information being the end point of a Wiener process under the original measure with and without a noise, and a more general one, where this extra information could be represented as an Itô integral of a deterministic process. I will also show an alternative way to calculate the first example using Malliavin derivative.
Chapter II

Related literature

There are many papers dealing with the topic of insider trading and its mathematical
modelling since the early 1990s even until these days. Karatzas et al. (1991), Pikovsky and Karatzas (1996) and Elliott et al. (1997) inspired and provided
a theory for complete markets on which the paper of Amendinger et al. (1998) is
based that I am going to introduce in further details and adapt at the end of this paper.
Results of Jacod (1985) and Follmer and Imkeller (1993) on initial enlargement of
filtrations and requirements for preserving semimartingale properties and no-arbitrage
gave a tool to deal with this problem from a new aspect. This requirement is that the
conditional law of the insider’s additional information must be absolutely continuous
to the regular law of this additional information. A general case of enlargement of
filtrations was studied by Jeulin (1980), but only the above mentioned criteria made
the theory applicable for this problem.

Imkeller et al. (2001) are examining cases where the above mentioned requirement
is not satisfied and enlargement of filtrations cannot be used straight away. There-
fore they use Itô and Malliavin calculus to find new conditions that would assure the
semimartingale property. They also examine the existence of arbitrage and show requ-
isites on the drift and volatility of the price process, which make it possible to give
exact strategies to utilize arbitrage possibilities. Duffie and Huang (1986) provided
the martingale theory used for this approach. Results of Denis et al. (2000) and
Grorud and Pontier (1998) gave a theoretical background for the use of Malliavin
calculus. For the arbitrage possibilities this framework deals with, it is necessary to be
familiar with the free lunches with vanishing risk theory introduced in Delbaen and
Schachermayer (1994).

In Imkeller (2003), it is shown that the drift present under the filtration of the insider
(because of which it is not a martingale under that filtration) is the key, when examining
arbitrage possibilities. This quantity could be grabbed by logarithmic Malliavin trace
of conditional distributions of the additional information with respect to the basic
filtration.

Acciaio et al. (2016) introduced another approach to deal with insider information
that is independent of model assumptions. They only examined cases where prices of
vanilla call options on an asset would represent the additional information. In their work they used Skorokhod embedding approach from Hobson (1998, 2011). Their framework adapted the one introduced in Beiglboeck et al. (2016), which provided the geometric properties of optimal models.
Chapter III

Theoretical framework

1 The market

Before dealing with the main question, let me present the market first. In this section, assumptions on the market participants and the investment opportunities will be introduced. It will be shown, how informed the investors are and what utility function reflects their preferences. To be able to deal with the enlarged filtration of the insider, Jacod’s results will be presented.

1.1 Information of the investors

As a base scenario, as in Amendinger et al. (1998), consider two investors: a regular one and an insider. The regular investor is well informed, but is only aware of all the public information available on the market. The other person is in possession of additional information that is not available for other market participants, therefore is called an insider.

Additional information is represented by a random variable $G$ which can be defined in several ways. For example, the insider might know the price of an asset in advance for a fixed time $T$, or the range in which the future price will be with or without some noise at time $T$. It could also be the value of some factor that has an impact on the price of a financial instrument for instance, or the insider may also know the maximum value of the price over the trading period. That is to say, given a probability space $(\Omega, \mathcal{F}, P)$ with the basic filtration, the ordinary investor makes decisions based on this complete and right-continuous filtration:

$$\mathcal{F} = (\mathcal{F}_t)_{t \in [0,T]} ,$$

while the insider can make decisions based on an enlarged filtration:

$$\mathcal{G} = (\mathcal{G}_t)_{t \in [0,T]} ,$$
where $\mathcal{G}_t = \bigcap_{\varepsilon>0} (\mathcal{F}_{t+\varepsilon} \vee \sigma(G))$, $G$ being a random variable that is $\mathcal{F}$-measurable taking values from the real space $\mathbb{R}$. Choosing $G$ this way has the beneficial property, that the conditional distribution of $G$ given $\mathcal{F}_t$ denoted as $P[G \in B|\mathcal{F}_t](\omega)$ has a regular version for all $t \in [0,T]$. This means that the random variables $P[G \in B|\mathcal{F}_t]$, (where $B \in \mathcal{B}(\mathbb{R})$) can be defined in such a way that the mapping $B \mapsto P[G \in B|\mathcal{F}_t](\omega)$ is a probability measure on the real line, for each $\omega$. In what follows $P[G \in B|\mathcal{F}_t]$ always refers to the regular version.

Defining the filtrations this way is equivalent to the statement above: the insider knows every information about the market that other investors know, but is also aware of some additional information.

### 1.2 Financial instruments available on the market

The investment opportunities on the market can be represented by a $d$-dimensional vector of the discounted prices $X = (X_{t,1}, X_{t,2}, \ldots, X_{t,d})'_{t \in [0,T]}$ of financial instruments. For the further examinations, let $M = (M_{t,1}, M_{t,2}, \ldots, M_{t,d})'_{t \in [0,T]}$ denote a continuous $d$-dimensional local $\mathbb{F}$-martingale and $\alpha = (\alpha_{t,1}, \alpha_{t,2}, \ldots, \alpha_{t,d})'_{t \in [0,T]}$ be a predictable $d$-dimensional process. We also make the technical assumption that $\alpha$ satisfies the following integrability condition

$$E \left( \int_0^T \alpha_t' d[M]_t \alpha_t \right) < \infty,$$

the necessity of which will be explained later.

We assume that the price process satisfies

$$dX_{t,i} = X_{t,i} \left( dM_{t,i} + \sum_{j=1}^d \alpha_{t,j} d[M_i,M_j]_t \right), \ i = 1, \ldots, d \ (1)$$

$$X_{0,i} > 0, \ \forall i,$$

where $M$ is a local martingale in $\mathbb{F}$, $\alpha$ satisfies the integrability condition above without which there would be no equivalent local martingale measure for $X$. The second statement is a natural assumption, as prices are always positive values.
1.3 Jacod’s condition and its application

Before moving forward with the introduction of the market, let me introduce the theorems presented in JACOD (1985) that gives a convenient way to deal with the above mentioned enlarged filtration: $G_t = \bigcap_{\varepsilon>0} (\mathcal{F}_{t+\varepsilon} \vee \sigma(G))$.

1.3.1 Introduction of two measures under the product space

The information that is in possession of the insider may be considered as a value from the product space $\Omega \otimes \mathbb{R}$, as for every random state $\omega \in \Omega$, the insider also knows the value of $G(\omega) \in \mathbb{R}$. The main idea was to consider the product space $\Omega \otimes \mathbb{R}$, with the product measure $\tilde{P} = P \otimes \mathcal{P}[G \in \cdot]$, where $P[G \in \cdot]$ is the distribution of $G$. Under the product measure the two coordinates are independent. That is to say, if $\pi_1, \pi_2$ denotes the two coordinate mappings on $\Omega \otimes \mathbb{R}$, where $\pi_1(\omega, x) = \omega$ and $\pi_2(\omega, x) = x$, then $\pi_1$ and $\pi_2$ are independent. This means, if we choose any set $A \subset \Omega$ and $B \in \mathcal{B}(\mathbb{R})$, then

$$\tilde{P}(\pi_1 \in A \times \pi_2 \in B) = P(A) \times P(G \in B) = \tilde{P}(\pi_1 \in A) \tilde{P}(\pi_2 \in B).$$

We introduce another measure $\hat{P}$ on the product space, which is the law of $\omega \mapsto (\omega, G(\omega))$, meaning $\hat{P}(A \times B) = P(A \cap (G \in B))$. Then we have

$$\hat{P}(A \times B) = P((\omega \in A) \cap (G \in B)) = E(\chi_A P[G \in B | \mathcal{F}_t](\omega)) \text{ for } A \in \mathcal{F}_t. \quad (2)$$

1.3.2 The relationship of the two measures

In order to find the connection between $\hat{P}$ and $\tilde{P}$, let’s suppose that Jacod’s condition holds, that is, the conditional law of $G$ given $\mathcal{F}_t$ is absolutely continuous with respect to the law of $G$. Denote the $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$ measurable function by $p_t$. To switch to the product measure we would need to write the conditional law of $G$ in a form $P[G \in B | \mathcal{F}_t](\omega) = \int_B p_t(\omega, x) P[G \in \cdot](dx)$, with the appropriate choice of $p_t(\omega, x)$ and $P[G \in \cdot]$ denoting the law of $G$, so that Equation (2) would further equal to

$$\hat{P}(A \times B) = E\left(\chi_A \int_B p_t(\omega, x) P[G \in \cdot](dx)\right) = \int_A \int_B p_t(\omega, x) P[G \in \cdot](dx) P(d\omega) \text{ for } A \in \mathcal{F}_t.$$
This form enables us to switch to the product measure as the following equation holds for (2)

$$\int_{A \times B} p_t(x, \omega) P [G \in .] (dx) P(d\omega) = \int_{A \times B} p_t(x, \omega) \tilde{P}(d\omega \times dx).$$

As a conclusion we obtained that, when Jacod’s condition holds, $\hat{P}$ is absolutely continuous with respect to $\tilde{P}$ on the $\sigma$-algebra $\tilde{G}_t$ generated by the sets \{A $\times$ B : A $\in$ $\mathcal{F}_t$, B $\in$ $\mathcal{B}(\mathbb{R})$\} and the density process is given by $(p_t)_{t \in [0, T]}$,

$$\hat{P}|_{\tilde{G}_t} \ll \tilde{P}|_{\tilde{G}_t} \text{ at any time } t < T.$$

The beneficial property of the product measure could be utilized, if we introduce the operation $\tilde{G} : \Omega$ $\otimes$ $\mathbb{R}$ $\mapsto$ $\mathbb{R}$, which takes the second coordinate of a variable on the product space $\tilde{G}(\omega, x) = x$. By definition $\tilde{G}$ is $\tilde{G}_0$ measurable. So the insider information represented by this second coordinate is part of the initial filtration of the insider, but it is not known to the regular trader.

Applying Girsanov theorem we get that a Brownian motion $\tilde{W}$ in $\tilde{G}_t$ under the product measure $\tilde{P}$ defines a Brownian motion in $\tilde{G}_t$ under the measure $\hat{P}$

$$\hat{W} = \tilde{W} - \left[ \tilde{W}, \mathcal{M} \right],$$

where $\mathcal{M}$ is the martingale from the density process $p_t(x, \omega) = \exp \left\{ \mathcal{M}_t - \frac{1}{2} \left[ \mathcal{M} \right]_t \right\}$.

1.3.3 Conditions for the equivalence of measures

If we only consider $\Omega$ instead of the product space $\Omega$ $\otimes$ $\mathbb{R}$, absolute continuity is not enough. In this case we require the equivalence of the two measures on $\tilde{G}_t$, for which further conditions need to be satisfied. As the density function is a non-negative martingale for all $\ell$ $\in$ $\mathbb{R}$, we can choose it to be a $\Omega$ $\times$ $\mathbb{R}$ $\times$ $[0, T)$-measurable, right-continuous function $p_t(\omega, \ell)$ with left-limits in $t$. Defining the stopping time $\tau$ as the first time the left-limit of $p(\cdot, \ell)$ hits zero or $T$, if it never reaches zero: $\tau(\ell) = \inf \{ t \geq 0, p_{t-}(\ell) = 0 \} \wedge T$, both $p(\cdot, \ell)$ and its left-limits $p_-(\cdot, \ell)$ are strictly positive in the stochastic interval $[0, \tau(\ell)]$, and zero once they hit zero: $p(\cdot, \ell) = 0$ on $[\tau(\ell), T]$. This process $P$-almost
surely does not reach zero before $T$, that is $\tau(G) = T$, provided that $\hat{P}$ and $\tilde{P}$ are equivalent on $\tilde{G}_t$ for all $t < T$.

If we use that $P$-almost surely $\tau(G) = T$, we can define the process $\frac{1}{\rho_t(\omega, G)}$. This is a positive $\mathcal{G}$-martingale, if the regular conditional distribution of $G$ given $\mathcal{F}_t$ is indeed not only absolutely continuous, but is also equivalent to the law of $G$. This process would therefore be the Radon-Nikodym derivative of the change of measures in the other way round. It defines measure $\tilde{P}_t$ on $(\Omega, \mathcal{G}_t)$, which is very useful as on $\mathcal{F}_t$ it is the same as $P$, but the $\sigma$-algebra of $\mathcal{F}_t$ becomes independent of $\sigma(G)$ under this new probability measure. This measure preserves martingale properties under the enlarged filtration: any (local) $F$-martingale under the original measure would be a (local) $\mathcal{G}$-martingale and also a (local) $F$-martingale under the new measure, therefore it is often referred to as $[0, t]$-insider martingale measure.

1.3.4 Jacod’s condition

For the rest of the sections in Chapter III, because of the above reasons, we need to assume that the following condition holds for the conditional law of $G$ on the product space $\Omega \otimes \mathbb{R}$ to be able to deal with the enlarged filtration using Jacod’s framework.

$$P[G \in \cdot | \mathcal{F}_t](\omega) \ll P[G \in \cdot] \quad \text{for all } t \in [0, T) \text{ and } P\text{-almost all } \omega \in \Omega.$$ 

Note that we only require the absolute continuity for $t < T$. For $t = T$, as $G$ is $F_T$ measurable, such relation cannot hold unless $G$ is discrete.

When instead of the product space, we are looking at $\Omega$, this condition is restricted to

$$P[G \in \cdot | \mathcal{F}_t](\omega) \text{ is equivalent to } P[G \in \cdot] \quad \text{for all } t \in [0, T) \text{ and } P\text{-almost all } \omega \in \Omega.$$ 

1.4 Application of the above results to the market

As a result of the change of measures there exist an $m = (m_t(\omega, \ell))$ process, measurable under the predictable $\sigma$-algebra on $\Omega \otimes \mathbb{R} \otimes [0, T)$ denoted by $\mathcal{P}(\hat{\mathcal{F}}) = \mathcal{P}(\mathcal{F}) \otimes \mathcal{B}(\mathbb{R})$,
where $\hat{F} := \{\hat{F}_t\}_{t \in [0,T)}$ with $\hat{F}_t := \bigcap_{\varepsilon > 0} (\mathcal{F}_{t+\varepsilon} \otimes \mathcal{B}(\mathbb{R}))$ (hat denoting the appropriate product space), that satisfies:

$$[p(\ell), M_i] = \int (m(\ell))_i p(\ell) d[M_i]_s,$$

and $\int_0^t |(m_t(G))_i d[M_i]_s < \infty$, which gives a canonical decomposition for these $\mathcal{G}$-semimartingales on the $[0, T)$ interval, which is the following form in our case:

$$\hat{M}_{t,i} = M_{t,i} - \int_0^t (m_s(G))_i d[M_i]_s, i = 1, \ldots, d,$$

where $\hat{M}$ denotes a local martingale under $\mathcal{G}$.

Given that $M$ is a semimartingale under $\mathcal{G}$ in $[0, T)$, a lemma introduced in Amendinger et al. (1998) is applicable. It states that there exists a $(\mu_t(\ell))$ process that is measurable under the product $\sigma$-algebra $\mathcal{P}(\hat{F})$, that satisfies the following equation:

$$
\begin{bmatrix}
\int_0^t (m_s(\ell))_1 d[M_1]_s \\
\vdots \\
\int_0^t (m_s(\ell))_d d[M_d]_s
\end{bmatrix}
= \int_0^t d[M]_s \mu_s(\ell), \text{ for any } \ell \in U, t < T.
$$

Therefore the canonical decomposition of the $\mathcal{G}$-local martingale simplifies to the following form:

$$\hat{M} = M - \int d[M] \mu(G).$$

This is equivalent to $d\hat{M}_i = dM_i - (m(G))_i d[M_i]$, from where

$$dM_i = d\hat{M}_i + (m(G))_i d[M_i].$$

Replacing $dM_i$ in the formula of the discounted price process (1) gives the following expression:

$$dX_i = X_i \left( d\hat{M}_i + (m(G))_i d[M_i] + \sum_{j=1}^d \alpha_{ij} d[M_i, M_j] \right),$$

13
and because of the lemma, this equals forward to
\[ dX_i = X_i \left( d\tilde{M}_i + (d[M] (\alpha + \mu (G)))_i \right). \]

As mentioned above, instead of looking at the closed interval, we only dealt with \([0, T)\). A natural way to extend it to \([0, T]\) is to define \(\tilde{M}_T := \lim_{t \to T} \tilde{M}_t\). This extension can be done, as a continuous local martingale is convergent if and only if its quadratic variation does not approach infinity, if \(t \to T\). \(M\) and \(\tilde{M}\) only differ in a bounded variation process, their quadratic variation \([M] = [\tilde{M}]\) is the same and we know \([M]_t\) does not approach infinity, if \(t \to T\).

It is guaranteed that there are no immediate arbitrage possibilities by the next condition:
\[ E \left( \int_0^T (\mu_s (G))' d[M]_s (\mu_s (G)) \right) < \infty, \]
for further details see also DELBAEN AND SCHACHERMAYER (1995). According to IMKELLER ET AL. (2001), Amendinger proved that there are no arbitrage possibilities when the conditional law of the additional information \(G\) is equivalent to the law of \(G\). They also present a few conditions on the drift and the volatility of the price process for which they could construct arbitrage strategies, but these are not subject of this thesis. In the following we shall call \((\mu_t (G))_{t \in [0, T]}\) the information drift of \(G\), as it will play a crucial role.

## 2 The strategy

After describing the market, let me focus on the strategy of both investors, which will define their discounted wealth process \((V_t(x, \pi))_{t \in [0, T]}\) as in (AMENDINGER ET AL., 1998). As stated before, the two investors make their decisions based on different information sets, which also means different filtrations. In order to make it easier to deal with the wealth process for both investors, let \(\mathbb{H} \in \{\mathbb{F}, G\}\) denote a generic filtration. On one hand, the discounted wealth process depends on the money they invest at the first place, that is to say
\[ V_0(x, \pi) = x. \]
On the other hand, it also depends on what strategy \( \pi = (\pi_s)_{s \in [0,t]} \in \mathbb{R}^d \) they choose to follow. The strategy consists of choosing what ratio of the total wealth to put in each asset for the subsequent period. It is obvious that they need to know how much they will invest in each security at the beginning of each period, they cannot make decisions retrospectively. Negative \( \pi \) value represents short selling. In mathematical terms \( \pi \) needs to be a \( \mathbb{H} \)-predictable process. It is also requested from \( \pi \) to satisfy the following condition for all \( t \in [0,T] \)

\[
\int_0^t \pi'_s d[M]_s \pi_s < \infty
\]

\( P \)-almost surely for the stochastic integral to be interpretable. Another condition for \( \pi \) is to be admissible with

\[ V_t(x, \pi) > 0 \text{ for all } t \leq T. \]

It is only interesting to investigate \( \pi \) in cases, where it is satisfied that

\[ E \left( \log^- V_T(x, \pi) \right) < \infty, \]

meaning that the expected terminal portfolio return is bounded from below. The dynamics of the discounted wealth process using the self-financing condition

\[
dV_s(x, \pi) = \sum_{i=1}^d \pi_{s,i} V_s(x, \pi) \frac{dX_{s,i}}{X_{s,i}}, \quad s \in [0, t]
\]

and \( \mathcal{E} \) denoting the Doléans-exponential, can be written as

\[
V_s(x, \pi) = x \mathcal{E} \left( \int \sum_{i=1}^d \pi_{z,i} \frac{dX_{z,i}}{X_{z,i}} \right) = x \mathcal{E} \left( \int \pi'_z dM_z + \int \pi'_z d[M]_z \alpha_z \right)_s,
\]

whereas under the insider insider measure instead of \( dM_z \) and \( \alpha_z \), it is \( d\tilde{M}_z \) and \( \alpha_z + \mu_z(G) \) in the equation respectively

\[
V_s(x, \pi) = x \mathcal{E} \left( \int \pi'_z d\tilde{M}_z + \int \pi'_z d[M]_z (\alpha_z + \mu_z(G)) \right)_s. \quad (3)
\]

Based on these, the insider can choose the strategy \( \pi \) from a wider set compared to the regular trader.
3 Utility maximization

The utility of each investor is per assumption the expected logarithmic value of the discounted wealth process in $T$

$$E(\log V_T(x, \pi)).$$

It looks the same for both of them, but the content behind is different, as their strategy is not identical. The regular investor makes all decisions and chooses a strategy based on the basic filtration $\mathcal{F}$, while the insider can utilize additional information on $\mathcal{G}$, when choosing a strategy, that is to say the insider’s strategy is $\mathcal{G}$-predictable.

We assumed the utility to be logarithmic to utilize the exponential properties of the discounted wealth process. Although it is not unrealistic, we could define a utility function that could capture the investors’ preferences more accurately (Amendinger et al., 1998). In the following two subsections, further critics and alternative function forms will be shown. Afterwards we get back to the utility maximization in a special and the generic case.

3.1 Motives and criticism of using logarithmic utility function

At first Bernoulli (1738) used the logarithmic form of the utility function, as he wanted to express utility as an increasing concave function, but no axiomatization was made about the investors’ behaviour. Maximizing expected logarithmic utility later turned out to be equivalent to maximizing the growth in the value of an investment in long-sequence, which made it even more popular to use it as a form of the utility function (Kardaras, 2010). It is important to note, that it would only lead to maximized final expected utility in long-sequence, because of the central limit theorem or the law of large numbers. However, studies have shown that following this strategy on finite number of periods can result in a utility far from the optimum. Furthermore, it is also shown that any non-uniform strategy (a strategy that changes depending on state of the wealth process) would be non-optimal (Samuelson, 1971).

3.2 Alternative utility functions

It is usually expected from utility functions to reflect the behaviour of a rational investor. It is also important that the utility maximization problems should have
closed form solutions, which enables us to use the theoretical results (Brockett and Golden, 1987). As a minimum requirement, every utility function needs to be complete, which in our case means it has to be able to identify which state of wealth process is as good as another for all the possible states. Reflexivity is another expectation, which means that every state is as good as itself. And lastly it has to be transitive, if \( x \) is as good as \( y \), and \( y \) is as good as \( z \), then \( x \) must be as good as \( z \) (Ingersoll, 2014).

HARA (hyperbolic absolute risk aversion) or LRT (linear risk tolerance) utility functions are commonly used, as they satisfy these minimum requirements. HARA functions are of the following form

\[
    u(w) = \frac{1 - \gamma}{\gamma} \left( \frac{a \cdot w}{1 - \gamma} + b \right)^\gamma,
\]

where \( b > 0 \) and \( \frac{a \cdot w}{1 - \gamma} + b > 0 \), so there is a lower bound for \( \gamma < 1 \) and an upper bound for \( \gamma > 1 \). For \( \gamma \) with higher integer values than one the wealth processes above the upper bound would be taken into account in the utility function, but their marginal utility would be negative. And the regarding LRT functions could be defined as

\[
    t(w) = \frac{w}{1 - \gamma} + \frac{b}{a},
\]

due to the fact that risk-tolerance functions are the reciprocal of risk-aversions, which are \(-\frac{u''(w)}{u'(w)}\).

The special cases of these functions are commonly used. First of all, namely the linear version, when \( \gamma \) is chosen to be one. This would represent the risk neutral case, which is not realistic. Isoelastic, which is also known as power utility is also a special case of this function, where \( \gamma < 1 \) and \( b = 0 \). And the well known logarithmic utility is attained when choosing both \( \gamma \) and \( b \) to be zero. Both isoelastic and logarithmic case has the speciality that their relative risk aversion is constant, which also means that their absolute risk aversion would increase. Negative exponential utility functions are: \( u(w) = -\exp(-a \cdot w) \), which have the property of constant absolute risk aversion that is unrealistic as not all investors have the same risk appetite (Ingersoll, 2014).

As a conclusion none of the well-known and commonly used utility functions have better properties then logarithmic utility function.
3.3 A special case of utility maximization

In this subsection, the restricted case introduced in Amendinger et al. (1998) will be shown, but in more detail than in the original version. This would give the base logic behind the optimization. If we take a look at a more special case, where our previous assumption for all \( t \leq T \) being

\[
\int_0^t \pi_s' d[M]_s \pi_s < \infty
\]

is restricted to

\[
E \left( \int_0^T \pi_s' d[M]_s \pi_s \right) < \infty,
\]

then \( \int_0^T \pi_s' d\tilde{M}_s \) and \( \int_0^T \pi_s' dM_s \) will become true martingales. Because of Equation (3), our optimization problem for the insider would mean the maximization of the expected value of the following expression

\[
\log V_T(x, \pi) = \log \left( x \mathcal{E} \left( \int \pi_z' d\tilde{M}_z + \int \pi_z' d[M]_z (\alpha_z + \mu_z (G)) \right) \right) = (4)
\]

\[
= \log x + \int_0^T \pi_z' d\tilde{M}_z + \int_0^T \pi_z' d[M]_z (\alpha_z + \mu_z (G)) - \frac{1}{2} \int_0^T \pi_z' d[M]_z (\alpha_z + \mu_z (G)) - \frac{1}{2} \pi_z
\]

as \( M \) and \( \tilde{M} \) only differ in a bounded variation process, their quadratic variation \( d[M]_z = d[\tilde{M}]_z \) is the same. This could be further expanded by adding and subtracting the same term \( \frac{1}{2} \int_0^T (\alpha_z + \mu_z (G))' d[M]_z (\alpha_z + \mu_z (G)) \), Equation (4) becomes

\[
\log V_T(x, \pi) = \log x + \int_0^T \pi_z' d\tilde{M}_z + \frac{1}{2} \int_0^T (\alpha_z + \mu_z (G))' d[M]_z (\alpha_z + \mu_z (G)) -
\]

\[
- \frac{1}{2} \int_0^T (\alpha_z + \mu_z (G))' d[M]_z (\alpha_z + \mu_z (G)) + \int_0^T \pi_z' d[M]_z \left( \alpha_z + \mu_z (G) - \frac{1}{2} \pi_z \right).
\]
The last two terms of which could be written as
\[-\frac{1}{2} \int_0^T (\alpha_z + \mu_z(G))'d[M]_z(\alpha_z + \mu_z(G)) + \int_0^T \pi'_zd[M]_z(\alpha_z + \mu_z(G) - \frac{1}{2}\pi_z) = (5)\]

\[= \int_0^T -\frac{1}{2}\left(\alpha'_z d[M]_z \alpha_z + \alpha'_z d[M]_z \mu_z(G) + (\mu_z(G))'d[M]_z \alpha_z + (\mu_z(G))'d[M]_z \mu_z(G) + \right.\]
\[\left. + \int_0^T \left(\pi'_z d[M]_z \alpha_z + \pi'_z d[M]_z \mu_z(G) - \frac{1}{2}\pi'_z d[M]_z \pi_z\right)\]

after expanding the terms inside the integral, applying the constant factor rule and using the linearity of integration. The latter assures that the sum of the two integrals could be interpreted as one integral with the terms in the body of the integral summed. In the next step the two integrals above are joint and using that \(d[M]_z\) is a symmetric \(d \times d\) sized matrix by definition (Karatzas and Shreve (1998)), the following transformation could be made \(\int_0^T \pi'_zd[M]_z \alpha_z = \int_0^T -\frac{1}{2} (-2) (\pi'_z d[M]_z \alpha_z) = \int_0^T -\frac{1}{2} (-\pi'_z d[M]_z \alpha_z - \alpha'_z d[M]_z \pi_z)\), and the same could be done to \(\int_0^T \pi'_zd[M]_z \mu_z(G)\) similarly. As a result we would get that Equation (5) is
\[\int_0^T -\frac{1}{2}\left(\alpha'_z d[M]_z \alpha_z + \alpha'_z d[M]_z \mu_z(G) + (\mu_z(G))'d[M]_z \alpha_z + (\mu_z(G))'d[M]_z \mu_z(G) - \right.\]
\[\left. -\pi'_z d[M]_z \alpha_z - \alpha'_z d[M]_z \pi_z - \pi'_z d[M]_z \mu_z(G) - (\mu_z(G))'d[M]_z \pi_z + \pi'_z d[M]_z \pi_z\right) =
\[= -\frac{1}{2} \int_0^T (\alpha_z + \mu_z(G) - \pi_z)'d[M]_z(\alpha_z + \mu_z(G) - \pi_z)
\]

Using this result enables us to write the logarithmic utility from Equation (4) as
\[\log V_T(x, \pi) = \log x + \int_0^T \pi'_zd\tilde{M}_z + \frac{1}{2} \int_0^T (\alpha_z + \mu_z(G))'d[M]_z(\alpha_z + \mu_z(G)) - \]

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\[-\frac{1}{2} \int_0^T (\alpha_z + \mu_z(G) - \pi_z)\,d[M]_z (\alpha_z + \mu_z(G) - \pi_z).\]

The first and the third terms of the equation are constants with respect to \(\pi\), so their expected values are not affected by the choice of \(\pi\). The second one is a martingale because of the special assumption before, the expected value of which is zero by definition. To maximize the expected value of the whole expression, we should focus on the last term: we need to minimize the value we are subtracting, which could be attained by choosing

\[\pi_z = \alpha_z + \mu_z(G)\text{ for every } z \leq T.\]  

This could be done as by definition \(\alpha\) is a predictable process, \(\mu(G)\) is measurable in the product \(\sigma\)-algebra, therefore also progressively measurable in \(\mathcal{G}\), and because of the integrability assumptions on both processes, the strategy will be admissible with respect to the filtration \(\mathcal{G}\). The maximized utility up to time \(T\) would be therefore

\[\log x + \frac{1}{2} E \left( \int_0^T (\alpha_z + \mu_z(G))\,d[M]_z (\alpha_z + \mu_z(G)) \right).\]

### 3.3.1 Optimal utility of the regular investor

The maximization problem for the regular investor following the same logic

\[
\log V_T(x, \pi) = \log \left( x e^{\left( \int \pi_z' dM_z + \int \pi_z' [M]_z \pi_z \right)} \right) =
\]

\[= \log x + \int_0^T \pi_z' dM_z + \frac{1}{2} \int_0^T \alpha_z' [M]_z \alpha_z - \frac{1}{2} \int_0^T (\alpha_z - \pi_z)' [M]_z (\alpha_z - \pi_z),\]

from which the \(\mathbb{F}\)-predictable process \(\pi_z = \alpha_z\) will be the optimal strategy for every \(z \leq T\) with

\[\log x + \frac{1}{2} E \left( \int_0^T \alpha_z' [M]_z \alpha_z \right)\]

being the utility in optimum until time \(T\).
3.3.2 Additional logarithmic utility of the insider

The difference between the two utilities up to time \( T \) would give the additional expected logarithmic utility gain of the insider for this special case

\[
\frac{1}{2} E \left( \int_0^T \left( \alpha_z + \mu_z (G) \right)' d[M]_t \left( \alpha_z + \mu_z (G) \right) - \int_0^T \alpha_z' d[M]_t \alpha_z \right) =
\]

\[
= \frac{1}{2} E \left( \int_0^T \alpha_z' d[M]_t \mu_z (G) + \int_0^T \mu_z' (G) d[M]_t \left( \alpha_z + \mu_z (G) \right) \right) =
\]

\[
= E \left( \int_0^T \alpha_z' d[M]_t \mu_z (G) + \frac{1}{2} \int_0^T \mu_z' (G) d[M]_t \mu_z (G) \right).
\]

3.4 General case of utility maximization

In the generic case, we do not restrict the strategies of the investors with the assumption made in the beginning of the previous section anymore. That is, we only require

\[
\int_0^t \pi_s' d[M]_t \pi_s < \infty
\]

but not necessarily

\[
E \left( \int_0^t \pi_s' d[M]_t \pi_s \right) < \infty.
\]

Nevertheless, the results could be given by the same form, which will be proven in this section. As shown before, we could also look at the regular agent’s optimization problem as a restricted case of the insider’s, only with \( \mu_z (G) \equiv 0 \) and every other process interpreted on the base filtration. Therefore we will only deal with the insider’s maximization problem.

As presented in Amendinger et al. (1998), the well-known inequality

\[
k(a) \leq k \left( (k')^{-1} (b) \right) - b \left( (k')^{-1} (b) - a \right), \forall a, b,
\]

(7)
where $k$ is a concave $C^1$-function $(k')^{-1}(b)$ is the value of the inverse of the first derivative of $k$ in $b$, could be used to prove that the solution for the special case has identical form to the one shown in the previous section

$$\pi_z = \alpha_z + \mu_z(G) \text{ for every } z \leq T.$$ 

In our case the concave function $k(.) = \log(.)$, $a$ means the value of the discounted wealth process in time $T$, and let $b = yZ_T$, where $y > 0$ is a constant and $Z_T = \mathcal{E}\left(-\int (\alpha + \mu(G))'d\tilde{M}_t\right)_T$. Note that in the case of the regular trader one would need to integrate with regards of $dM_t$ instead of $d\tilde{M}_t$ in the definition of $Z_T$. By the definition of the inverse, we also know that $(k')^{-1}(b) = c \iff k'(c) = b$ for any $c$, so $(k')^{-1}(b)$ would give the place $c$, where the steepness of our original concave function $k$ is $b$. In our case as $k(.) = \log(.)$, the steepness $k'(.) = \log'(.) = \frac{1}{y}$ would only be $b$, if we would look at the function $k'(\frac{1}{b}) = b$, so $(k')^{-1}(b) = \frac{1}{b}$. Substituting everything in the above inequality (7), we get an upper bound for the logarithmic wealth process

$$\log(V_T(x, \pi)) \leq \log\left(\frac{1}{yZ_T}\right) - yZ_t\left(\frac{1}{yZ_T} - V_T(x, \pi)\right) =$$

$$= \log\left(\frac{1}{yZ_T}\right) - 1 + yZ_T V_T(x, \pi).$$

Taking the expectation of both sides gives, the expected logarithmic utility appears on the left hand side

$$E(\log(V_T(x, \pi))) \leq \log\left(\frac{1}{y}\right) - E(\log Z_T) - 1 + yE(Z_T V_T(x, \pi)). \quad (8)$$

First, to be able to calculate the value of $E(Z_T V_T(x, \pi))$, we need to get the dynamics of the process $Z_T V_T(x, \pi)$. We already know that according to the Equation (3) the dynamics of the discounted wealth process could be given by

$$dV_t(x, \pi) = V_t(x, \pi)\pi_t d\tilde{M}_t + V_t(x, \pi)\pi_t' d[M]_t \left(\alpha_t + \mu_t(G)\right),$$

while the dynamics of $Z$ are

$$dZ_t = -Z_t(\alpha_t + \mu_t(G))'d\tilde{M}_t.$$
Applying the well-known product rule \( dZ_t V_t = Z_t dV_t + V_t dZ_t + dZ_t V_t \) gives

\[
dZ_t V_t(x, \pi) = Z_t V_t(x, \pi) \pi_t' d\tilde{M}_t + Z_t V_t(x, \pi) \pi_t' [M]_t (\alpha_t + \mu_t (G)) -
\]

\[-Z_t V_t(x, \pi) (\alpha_t + \mu_t (G))' d\tilde{M}_t - Z_t V_t(x, \pi) \pi_t' [M]_t (\alpha_t + \mu_t (G)) =
\]

\[= Z_t V_t(x, \pi) \left( Z_t \pi_t' - (\alpha_t + \mu_t (G))' \right) d\tilde{M}_t,
\]

which means the process \( Z_t V_t(x, \pi) \) is also a local \( G \)-martingale, therefore a supermartingale as \( \tilde{M} \), but starting from \( x \). Because of the supermartingale property we get that \( E (Z_T V_T(x, \pi)) \leq x \). Getting back to the inequality (8), using this result gives

\[
E (\log(V_T(x, \pi))) \leq \log \frac{1}{y} - E (\log Z_T) - 1+yE (Z_T V_T(x, \pi)) \leq \log \frac{1}{y} - E (\log Z_T) - 1+yx
\]

and substituting the definition of the discounted wealth process \( V_t(x, \pi) \) and the definition of \( Z_t \), (8) results in the following form

\[
\log x + E \left( \int_0^T \pi_z' d\tilde{M}_z + \frac{1}{2} \int_0^T (\alpha_z + \mu_z (G))' [M]_z (\alpha_z + \mu_z (G)) \right) \leq
\]

\[
\log \frac{1}{y} + E \left( \int_0^T (\alpha_z + \mu_z (G))' d\tilde{M}_z + \frac{1}{2} \int_0^T (\alpha_z + \mu_z (G))' [M]_z (\alpha_z + \mu_z (G)) \right) - 1 + yx.
\]

To show that the chosen strategy is optimal, we need to transform the upper bound to a form that reflects the expected logarithmic utility with the strategy \( \pi_z = \alpha_z + \mu_z (G) \) for every \( z \leq T \). For that, we need to choose \( y \) in a way so that we would get \( \log x = \log \frac{1}{y} - 1 + yx \). Choosing \( y = \frac{1}{z} \) satisfies this condition. So we got that the expected value of the discounted wealth process in \( T \) defined by the strategy \( \pi_z = \alpha_z + \mu_z (G) \) for every \( z \leq T \) is maximal, as for any arbitrary strategy \( \pi \), we could not exceed this value.
Chapter IV

Alternative approach

The treatment of the initial enlargement described in chapter III was based on Jacod’s condition. However, there are several cases that could be relevant in practice, where the conditional law of $G$ is not absolutely continuous to the law of $G$. For example, when $G$ is the maximum of the price over our time horizon, it is known that Jacod’s condition fails to hold. Therefore, the motivation for an alternative approach was to find a weaker criteria to be able deal with these type of extensions and examine the additional utility of the insider and whether there are arbitrage possibilities for the one in possession of the extra information.

In the following sections to simplify the problem, let’s deal with the special case where $M = W$, so the martingale we examined before is a one-dimensional $\mathbb{F}$-Wiener process. This chapter is based on the results presented in Imkeller (2003) and Imkeller et al. (2001). First, we must introduce Malliavin’s calculus to give a criteria that ensures us the semimartingale preserving property. Then the Clark-Ocone formula will be presented, which gives a form that is similar to the martingale representation to make it easier to deal with the information drift and keep the Brownian motion $W$ a martingale.

4 Malliavin’s derivative

Suppose that $W$ is a Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$. A random variable $F$ is smooth if it can be written as

$$F = f(W_{t_1}, \ldots, W_{t_d}), \ t_1, \ldots t_d \in [0, T],$$

with an infinitely differentiable function $f$ with polynomial growth, that is,

$$f \in C^\infty_p (\mathbb{R}^d).$$
The Malliavin derivative of a smooth random variable is a process defined by the formula

\[(DF)_s = D_s F = D_s f (W_{t_1}, \ldots, W_{t_d}) = \sum_{i=1}^{d} \chi_{s \in [0,t_i]} \frac{\partial}{\partial x_i} f (W_{t_1}, \ldots, W_{t_d}),\]

for \(s \in [0, T]\) the sum of those partial derivatives of \(f\) where the value of the Brownian motion in the argument is taken at or after time \(s\). The derivative defined this way could be also interpreted as a random variable with values in \(L^2([0, T])\). If we repeat this derivation \(n\) times, we can get an \(n\)-parameter field

\[(D^{\otimes n} F) (\omega) \in L^2([0, T]^n),\]

its value in \((s_1, \ldots, s_n) \in [0, T]^n\) being

\[(D^{\otimes n}_{s_1,\ldots,s_n} F) (\omega) \in \mathbb{R}.\]

The completion of the set of smooth random variables with respect to the following norm

\[\|F\|_{p,n} = \|F\|_p + \sum_{i=1}^{n} E \left( \left( \int_0^T (D^{\otimes n}_{s_1,\ldots,s_n} F)^2 ds_1, \ldots, ds_i \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}\]

defines the Banach space \(D_{p,n}\), where \(n \in \mathbb{N}\) and \(p \geq 1\).

5 Clark-Ocone formula

For \(F \in D_{2,1}\) the Clark-Ocone formula gives the following representation

\[F = E (F) + \int_0^T E (D_s F | \mathcal{F}_s) dW_s.\]

The above expression could be applied to conditional densities that will make it easier to deal with them. The conditional density of \(G\) on \(\mathcal{F}_t\) was \(p_t (., \ell)\), the Radon-Nikodym derivative of the conditional law of \(G\) on \(\mathcal{F}_t\) with respect to the law of \(G\). By definition
the conditional law of $G$ is a martingale in the time parameter and for almost all $\ell \in \mathbb{R}$, and this martingale property is passed to the conditional density. Introducing

$$D_s p_s (\cdot, \ell) = \lim_{t \searrow s} D_s p_t (\cdot, \ell)$$

and utilizing that $p_t (\cdot, \ell)$ is a martingale in time parameter, the following holds

$$E(D_s p_t (\cdot, \ell) | \mathcal{F}_s) = D_s p_s (\cdot, \ell), \quad \text{and} \quad E(p_t (\cdot, \ell)) = p_0 (\cdot, \ell),$$

which implies that the Clark-Ocone formula in this case could be written in the following form

$$p_t (\cdot, \ell) = p_0 (\cdot, \ell) + \int_0^t D_s p_s (\cdot, \ell) dW_s, \quad t \in [0, T] \quad \text{and} \quad \ell \in \mathbb{R}.$$ 

### 6 The information drift

First, let’s suppose we have a Brownian motion $W$ on the probability space $(\Omega, \mathcal{F}, P)$, where the filtration is generated by $W$. Assuming Jacod’s condition still holds, we are going to calculate the information drift from the semimartingale decomposition. For that we shall find the connection between the density process and the information drift as in Imkeller (2003). We shall calculate the expected value of the drift of the $\mathcal{F}$-Wiener process between time $s$ and $t$ under the new measure, where the density process is the conditional density, $s \leq t$ and $s, t \in [0, T]$. For that let $A$ be any $\mathcal{F}_s$-measurable set and $B$ a Borel set with values from the real line. With these the typical set in a generator of $\mathcal{G}_s = \mathcal{F}_s \lor \sigma (G)$ could be written as $A \cap G^{-1} (B)$. So the value we were interested in could be written as

$$E(\chi_A \chi_B (G) (W_t - W_s)) = E \left( \int_B \chi_A (W_t - W_s) P [G \in d\ell | \mathcal{F}_t] \right).$$

(9)

Because instead of taking the expected value of the product of an expression and an indicator function of a set $B$, it is possible to integrate the expression on the set $B$ with regards to the applicable measure. Given the definition of $B$, the measure in this
case is the conditional law of $G$. Utilizing the definition of the conditional density, (9) further equals to

$$E \left( \int_B \chi_A (W_t - W_s) \, p_t (. , \ell) \, P \left[ G \in d\ell \right] \right).$$

As the calculation of expected value is actually an integration, if we apply Fubini’s theorem, we can change the sequence of the integrals

$$\int_B E \left( \chi_A (W_t - W_s) \, p_t (. , \ell) \right) \, P \left[ G \in d\ell \right].$$

It is possible to add a term $p_s (. , \ell)$ without modifying the value of Equation 9, given the fact that it is, among with $\chi_A$, $\mathcal{F}_s$-measurable, and the conditional expectation of the Wiener-increment is zero

$$\int_B E \left( \chi_A (W_t - W_s) \, (p_t (. , \ell) - p_s (. , \ell)) \right) \, P \left[ G \in d\ell \right].$$

We already know that the conditional density is a martingale in the time parameter, so the martingale representation could be applied to it with measurable kernels $k$

$$p_t (. , \ell) = p_0 (. , \ell) + \int_0^t k_u^\ell dW_u.$$ 

This implies that $p_t (. , \ell) - p_s (. , \ell)$ from our calculation expressed using kernels would be

$$p_t (. , \ell) - p_s (. , \ell) = \left( p_0 (. , \ell) + \int_0^t k_u^\ell dW_u \right) - \left( p_0 (. , \ell) + \int_0^s k_u^\ell dW_u \right) = \int_s^t k_u^\ell dW_u.$$ 

We also know that $\chi_A(W_t - W_s) = \int_0^t \chi_A \chi_{[s,t]}(u) \, dW_u$ and isometry implies that

$$E \left( \int_0^t \chi_A \chi_{[s,t]}(u) \, dW_u \cdot \int_0^t k_u^\ell dW_u \right) = E \left( \int_s^t \chi_A \cdot k_u^\ell du \right).$$ 

If we substitute these information in the formula (9) and apply a simple expansion, we get

$$E \left( \chi_A \chi_B (G) (W_t - W_s) \right) = \int_B E \left( \chi_A \int_s^t k_u^\ell du \right) \, P \left[ G \in d\ell \right] = \int_B \int_s^t k_u^\ell du \, P \left[ G \in d\ell \right].$$
As the conditional density is a martingale in time parameter for a fixed \( \ell \in \mathbb{R} \), it holds that \( E(p_t(\cdot,\ell)|\mathcal{F}_u) = p_u(\cdot,\ell) \). So the amount sought in Equation (9) further equals to

\[
\int_B E\left(\chi_A \int_s^t \frac{k_u^{\ell}}{p_u(\cdot,\ell)} p_u(\cdot,\ell) du\right) P[G \in d\ell],
\]

and applying Fubini’s theorem again

\[
= E\left(\int_B \chi_A \int_s^t \frac{k_u^{\ell}}{p_u(\cdot,\ell)} du \cdot p_t(\cdot,\ell) P[G \in d\ell]\right).
\]

If we look carefully, the already used relationship between the regular and conditional law of \( G \) appears again, so this would be equivalent to

\[
E\left(\int_B \chi_A \chi_B(G) \int_s^t \frac{k_u^{\ell}}{p_u(\cdot,\ell)} |_{\ell=G} du\right).
\]

and writing the integral on \( B \) back to its original indicator form, we get

\[
E\left(\chi_A \chi_B(G) \int_s^t \frac{k_u^{\ell}}{p_u(\cdot,\ell)} |_{\ell=G} du\right).
\]

So the information drift we wanted to determine is

\[
\mu_t(G) = \frac{k_t^{\ell}}{p_t(\cdot,\ell)} |_{\ell=G},
\]

where we know that \( k_t^{\ell} = D_t p_t(\cdot,\ell) \) by the Clark-Ocone formula, so it further equals

\[
\frac{D_t p_t(\cdot,\ell)}{p_t(\cdot,\ell)} |_{\ell=G},
\]

where if we take a closer look, we can find the rule for derivation of the log function:
\[ \log x = \frac{x'}{x}, \text{ so we can conclude that} \]
\[ \mu_t(G) = D_t \log p_t(\cdot, \ell)|_{\ell = G}, \]

the information drift can be written as the logarithmic Malliavin trace of the conditional density process.

This also means that by the definition of the information drift, a \( G \)-Brownian motion \( \bar{W}_t \) is a drifted Brownian motion under the original measure \( \bar{W}_t = W_t - \int_0^t D_s \log p_s(\cdot, G) ds, \) assuming \( D_t \log p_t(\cdot, \ell) \) is integrable: \( E \left( \int_0^T |D_t \log p_t(\cdot, \ell)| dt \right) < \infty. \)

7 Alternative condition for semimartingale preserving probability

So far only the case when Jacod’s condition held was examined. However, there is a weaker criteria on the semimartingale preserving property. To do so, let’s assume there exists a signed measure \( D_t P_t(dx) \) satisfying

\[ E(D_t \psi (G) | \mathcal{F}_t) = \int \psi (x) D_t P_t(dx). \] (10)

The new absolute continuity condition would be

\[ D_t P_t(dx) \ll P[G \in dx | \mathcal{F}_t] \ P\text{-almost surely for almost all } t \in [0, T]. \] (11)

Denote by \( q_t(\omega, x) \) the Radon-Nikodym derivative \( \frac{D_t P_t(dx)}{P[G \in dx | \mathcal{F}_t]} \) and suppose that for \( G \in D_{2,1} \) it satisfies the following

\[ E \left( \int_0^T |q_s(\cdot, G)| ds \right) < \infty. \] (12)
Then a $\mathcal{G}$-Brownian motion can be decomposed in the form

$$\tilde{W}_t = W_t - \int_0^T q_s(., G) ds,$$

(13)

where $W$ is an $\mathcal{F}$-Brownian motion.

### 7.1 Showing the sufficiency of the alternative condition

First let me examine the right-hand side of the Equation (13), if it really defines a $\mathcal{G}$-martingale. For that we need to check the statements of the Lévy characterization. The quadratic variation is indeed the time, as it is the same as the quadratic variation of $W$, given that they only differ in a bounded variation term. The starting point is known to be zero, as both $W_0 = 0$ and $\int_0^0 q_s(., G) ds = 0$. As $\mathcal{G}$ is larger than $\mathcal{F}$, measurability is also satisfied. It is only needed to check, whether it is a $\mathcal{G}$-local martingale.

Let $f$ be a smooth bounded function on $\mathbb{R}$, $\varepsilon > 0$ and $A \in \mathcal{F}_t$. To prove that $\tilde{W}$ is a $\mathcal{G}$-local martingale, we need to examine the below amount

$$E (\chi_A f (G) (W_{t+\varepsilon} - W_t)).$$

(14)

By assumption $f$ is smooth, so we can apply the Clark-Ocone formula for $f (G)$. As a result we get $f (G) = E(f (G)) + \int_0^T E(D_s f (G) | F_s) dW_s$, and we know that

$$W_{t+\varepsilon}-W_t = \int_0^T \chi_A \chi_{[t,t+\varepsilon]} dW_s.$$ The product $E(f (G)) (W_{t+\varepsilon} - W_t)$ is normally distributed with zero expectation, therefore as in the expression (14) we are taking the expectation this that term will be zero. For the other term we can apply isometry, from which we obtain that (14) further equals to

$$E \left( \chi_A \int_t^{t+\varepsilon} E (D_s f (G) | F_s) ds \right),$$

from where by the definition of $D_s P_x(dx)$ in (10), we get

$$E \left( \chi_A \int_t^{t+\varepsilon} \int_{\mathbb{R}} f (x) D_s P_x(dx) ds \right).$$
Now that the signed measure $D_s P_s(dx)$ appeared in the formula, we can apply the change of measures, leading (14) to be

$$E \left( \chi_A \int_t^{t+\varepsilon} \int_{\mathbb{R}} f(x) q_s(.,x) P[G \in dx|\mathcal{F}_s] \, ds \right).$$

We assumed $q(.,G)$ satisfies the integrability condition, therefore we can apply Fubini’s theorem, so (14) becomes

$$\int_t^{t+\varepsilon} E \left( \chi_A \int_{\mathbb{R}} f(x) q_s(.,x) P[G \in dx|\mathcal{F}_s] \right) \, ds.$$

Calculating the integral inside leads to

$$\int_t^{t+\varepsilon} E (\chi_A f(G) q_s(.,G)) \, ds,$$

from where applying Fubini’s theorem again, we can conclude that Equation (14) is

$$E \left( \chi_A f(G) \int_t^{t+\varepsilon} q_s(.,G) \, ds \right),$$

meaning that $\hat{W}$ is indeed a martingale in $\mathcal{G}$. Therefore it is shown that it is also $\mathcal{G}$-Brownian motion.

### 7.2 Alternative assumptions on $q(.,G)$

In this section, the results in Imkeller et al. (2001) will be shown without any proofs. The aim is only to illustrate the relationship between this alternative and our base framework. The new absolute continuity condition (11) with the integrability assumption (12) on $q(.,G)$ gives a weaker condition than Jacod’s. Therefore there might be cases, where $G$ only satisfies the assumption (11), but not Jacod’s condition presented in section (1.3.4). As a result, even if Jacod’s framework could not be applied, still these cases could be treated by enlargement of filtrations technique.

If we require $q(.,G)$ to be even square integrable, meaning $E \left( \int_0^T q_s^2(.,G) \, ds \right) < \infty$, we get the absolute continuity of the conditional law of $G$ to the regular law of $G$. This means that Jacod’s condition holds and then the results of Section 6 can be used.
However, in this case the equivalence of the conditional law of $G$ and the regular law of $G$ may still not hold. It is needed to make stronger assumptions for the equivalence. It is sufficient, if we restrict the integrability condition further to exponential integrability of the Novikov type, that is $E \left( \exp \left( \frac{1}{2} \int_0^T q_s^2 \left( \cdot, G \right) ds \right) \right) < \infty$. 
Chapter V

Application to simple examples

Let me apply the theories presented above for a few simple examples. In all of them, let’s consider a classical Black-Scholes market with one stock with Geometric Brownian Motion (GBM) dynamics

\[ dS_t = S_t (\mu dt + \sigma dW_t) \]

and a bond described by the following process

\[ dB_t = rB_t dt, \]

where \( \mu, \sigma \) and \( r \) are constants.

The wealth process (not the discounted one) of both investors would follow the dynamics below for the strategy to be self-financing (omitting the arguments of \( V_t(x, \pi) \) to be more easily readable)

\[ dV_t(x, \pi) = \pi_t \frac{V_t}{S_t} dS_t + (1 - \pi_t) \frac{V_t}{B_t} dB_t \]

replacing the dynamics of the stock and the bond, we get

\[ \pi_t V_t (\mu dt + \sigma dW_t) + (1 - \pi_t) V_t r dt = \]

\[ = V_t ((\pi_t \mu + (1 - \pi_t)r) dt + \pi_t \sigma dW_t) \]

for every \( t \). The solution of this stochastic differential equation in \( T \) gives the value of the wealth process at termination

\[ V_T(x, \pi) = x e^{\left( \int_0^T (\pi_t \mu + (1 - \pi_t)r) dt + \int_0^T \pi_t \sigma dW_t \right)} = \]

\[ = x e^{\left( \int_0^T (\pi_t (\mu - r) + r) dt + \int_0^T \pi_t \sigma dW_t \right)}. \]

If we denote the discounted wealth process by \( \tilde{V}_T(x, \pi) = \frac{V_T(x, \pi)}{B_T} \) the logarithmic utility
at time $T$ would be
\[
\log \tilde{V}_T(x, \pi) = \log x + \int_0^T \pi_t(\mu - r) dt + \int_0^T \pi_t \sigma dW_t + \frac{1}{2} \int_0^T \pi_t^2 \sigma^2 dt.
\]

And its expectation equals
\[
E \left( \log \tilde{V}_T(x, \pi) \right) = \log x + E \left( \int_0^T \left( \pi_t(\mu - r) - \frac{\pi_t^2 \sigma^2}{2} \right) dt + \int_0^T \pi_t \sigma dW_t \right) \tag{15}
\]

this is the amount that the investors wish to maximize in $\pi$, as that is the only parameter that is dependent on their choice.

In section 3.4, it was already shown that $\pi_t = \alpha_t + \mu_t (G)$ for every $t < T$ is optimal from the strategies that are square integrable on the product space. Based on Equation (15), $\alpha_t$ can be identified. Giving the optimal strategy for the regular trader to be
\[
\pi_t = \alpha_t = \frac{\mu - r}{\sigma^2} \text{ for every } t < T,
\]

which is indeed square integrable, it is even constant in time.

Substituting it back to the Equation (15), we get that in optimum the expected logarithmic utility of the regular investor is
\[
E \left( \log \tilde{V}_T(x, \pi) \right) = \log x + E \left( \int_0^T \left( \frac{\mu - r}{\sigma^2} (\mu - r) - \left( \frac{\mu - r}{\sigma^2} \right)^2 \sigma^2 \right) dt + \int_0^T \frac{\mu - r}{\sigma} dW_t \right) =
\]
\[
= \log x + E \left( \int_0^T \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 dt + \int_0^T \frac{\mu - r}{\sigma} dW_t \right).
\]

For optimization based on the enlarged filtration, it is needed to apply change of measures to know the properties of the $\mathbb{F}$-Brownian motion in the formula above under the insider measure, which will be done in the following sections.

Note, that in the above cases it was not used that $\mu$, $r$ and $\sigma$ are constants. It would be enough to only assume that these are adapted processes satisfying that
\[
\int_0^T \left( \frac{\mu(t) - r(t)}{\sigma^2(t)} \right)^2 dt < \infty.
\]
8 Insider information represented as $G = W_T$

Let me first consider the case when the insider knows the end point of a $\mathbb{F}$-Brownian motion. This also means that the insider is aware of the stock price in $T$, as the solution of the stochastic differential equation for $S_T$ only depends on deterministic values and $W_T$

$$S_T = S_0 \exp \left\{ \left( \mu - \frac{\sigma^2}{2} \right) T + \sigma W_T \right\}.$$

8.1 Finding the density process

It is needed to indentify the density process for the change of measures that we want to apply. For that as a first step, let’s calculate the conditional law of $G$, which is in this case

$$P[W_T \in dx | \mathcal{F}_t].$$

(16)

To utilize the independence of the Wiener increments, let’s write it in the following form

$$P[W_T - W_t + W_t \in dx | \mathcal{F}_t],$$

where $W_T - W_t$ is independent of $\mathcal{F}_t$, while $W_t$ is $\mathcal{F}_t$-measurable. Using the well known theorem we may calculate the above probability by fixing the measurable part as some constant $y$, and calculate the unconditional probability of the rest, then substitute $y$ with its original value

$$P[W_T - W_t + y \in dx | y=W_t].$$

It shows that the conditional law of $G$ given $\mathcal{F}_t$ is $N(y, T-t)$. Using the density function of the normal distribution, (16) further equals to

$$\frac{1}{\sqrt{2\pi(T-t)}} \exp \left\{ -\frac{(x-W_t)^2}{2(T-t)} \right\} dx.$$

As in the general theoretical introduction, we still would like to find the density process $p_t(., x)$ that would enable us to write this amount as a function of the unconditional probability of $G$

$$p_t(., x)P[W_T \in dx] = p_t(., x)\frac{1}{\sqrt{2\pi T}} \exp \left\{ -\frac{x^2}{2T} \right\} dx.$$
It could be easily seen that for these two lines to be equal, \( p_t(.,x) \) needs to be

\[
p_t(.,x) = \sqrt{\frac{T}{T-t}} \exp \left\{ -\frac{(x-W_t)^2}{2(T-t)} + \frac{x^2}{2T} \right\}. \tag{17}
\]

8.2 Writing the density process in Doléans-exponential form

To get the usual Doléans-exponential form of

\[
p_t(.,x) = \exp \left\{ \mathcal{M}_t - \frac{1}{2} [\mathcal{M}]_t \right\}
\]

that could be used in Girsanov theorem, let me apply Itô’s formula to \( h(t,Z) = \frac{(x-Z)^2}{T-t} \).

The derivatives are

\[
\frac{\partial}{\partial t} h = (x-Z)^2 \frac{-1}{(T-t)^2} = \frac{(x-Z)^2}{(T-t)^2}
\]

\[
\frac{\partial}{\partial Z} h = 2 \cdot \frac{x-Z}{T-t} (-1) = -\frac{2x}{T-t} + \frac{2Z}{T-t}
\]

\[
\frac{\partial^2}{\partial Z^2} h = \frac{2}{T-t}
\]

and Itô’s formula gives that the dynamics of \( h \) is

\[
\frac{dh(t,W_t)}{T-t} = \frac{(x-W_t)^2}{T-t} dt + \left( \frac{-2x}{T-t} + \frac{2W_t}{T-t} \right) dW_t + \frac{1}{2} \frac{2}{T-t} dt =
\]

\[
= \frac{(x-W_t)^2 + T-t}{(T-t)^2} dt + \frac{2W_t-2x}{T-t} dW_t.
\]

Writing this in integral form gives

\[
\frac{(x-W_t)^2}{T-t} = \frac{x}{T} + \int_0^t \frac{(x-W_s)^2 + T-s}{(T-s)^2} ds + \int_0^t \frac{2W_s-2x}{T-t} dW_s.
\]

Using the above after multiplying it with \(-\frac{1}{2}\), the term in the exponential of the expression \( p_t(.,x) = \sqrt{\frac{T}{T-t}} \exp \left\{ -\frac{(x-W_t)^2}{2(T-t)} + \frac{x^2}{2T} \right\} \) (17) can be modified to

\[
-\frac{(x-W_t)^2}{2(T-t)} + \frac{x^2}{2T} = -\frac{1}{2} \int_0^t \frac{(x-W_s)^2 + T-s}{(T-s)^2} ds + \int_0^t \frac{x-W_s}{T-s} dW_s.
\]
This means that the martingale term \( \mathcal{M}_t \) could only be \( \int_0^t \frac{x-W_s}{T-s} dW_s \), the quadratic variation of which is \( \int_0^t \left( \frac{x-W_s}{T-s} \right)^2 ds \) and the value at starting point is 0. However, in the above term, we still have \( -\frac{1}{2} \int_0^t \frac{T-s}{(T-s)^2} ds = \frac{1}{2} \int_0^t \frac{-1}{T-s} ds \) in the exponential, which is a deterministic integral and is easy to calculate: \( \frac{1}{2} \int_0^t \frac{-1}{T-s} ds = \frac{1}{2} \left[ \log (T-s) \right]_0^t = \frac{1}{2} \log (T-t) - \frac{1}{2} \log T = \log \sqrt{\frac{T-t}{T}} \). This omits the multiplier of the exponential in the expression of \( p_t(\cdot, x) \), so the martingale term is indeed \( \int_0^t \frac{x-W_s}{T-s} dW_s \), and

\[
p_t(\cdot, x) = e^\left( \int_0^t \frac{x-W_s}{T-s} dW_s \right). \]

8.3 Utility maximization

Applying Girsanov theorem, we get that every \( \mathcal{F} \)-Brownian motion \( W \) is a drifted Brownian motion in the enlarged filtration

\[
\tilde{W}_t = W_t - [W, \mathcal{M}]_t = W_t - \int_0^t \frac{W_T-W_s}{T-s} ds,
\]

as \( W \) and the \( \mathcal{G} \)-Brownian motion \( \tilde{W} \) are both Wiener processes, only differing in their drifts their quadratic covariation is \( ds \).

To identify the information drift, let’s recall what we got for the expected logarithmic utility function

\[
E \left( \log \tilde{V}_T(x, \pi) \right) = \log x + E \left( \int_0^T \left( \pi_t (\mu - r) - \pi_t^2 \frac{\sigma^2}{2} \right) dt + \int_0^T \pi_t \sigma d\tilde{W}_t \right).
\]

Utilizing that under the insider measure \( W_t = \tilde{W}_t + \int_0^t \frac{W_T-W_s}{T-s} ds \), we can substitute \( dW_t = d\tilde{W}_t + \frac{W_T-W_T}{T-t} dt \) that leads to

\[
E \left( \log \tilde{V}_T(x, \pi) \right) = \log x + E \left( \int_0^T \left( \pi_t \left( \mu - r + \sigma \frac{W_T-W_t}{T-t} \right) - \pi_t^2 \frac{\sigma^2}{2} \right) dt + \int_0^T \pi_t \sigma d\tilde{W}_t \right).
\]
From this form, the information drift from the semimartingale decomposition is given by
\[ \mu_t(G) = \frac{W_T - W_t}{\sigma (T - t)}. \]

Based on the results of Section 3.4, the optimal strategy for the insider would be
\[ \pi_t = \alpha_t + \mu_t(G) = \frac{\mu - r + \sigma \frac{W_T - W_t}{T - t}}{\sigma^2}, \]
which gives that the expected logarithmic utility of an insider is
\[
E \left( \log \tilde{V}_T(x, \pi) \right) = \log x + E \left( \int_0^T \left( \frac{\mu - r + \sigma \frac{W_T - W_t}{T - t}}{\sigma^2} \right)^2 \left( \frac{W_T - W_t}{T - t} \right)^2 \sigma d\tilde{W}_t \right) + E \left( \int_0^T \left( \frac{\mu - r + \sigma \frac{W_T - W_t}{T - t}}{\sigma^2} \right)^2 \frac{\sigma^2}{2} \right) dt.
\]

This means that the additional logarithmic utility of an insider is
\[
E \left( \int_0^T \left( \frac{\mu - r + \sigma \frac{W_T - W_t}{T - t}}{\sigma^2} \right)^2 \frac{\sigma^2}{2} \right) dt = \int_0^T \left( \frac{\mu - r + \sigma \frac{W_T - W_t}{T - t}}{\sigma^2} \right)^2 \frac{\sigma^2}{2} dt \]
and as \( (\mu - r + \sigma \frac{W_T - W_t}{T - t})^2 = (\mu - r)^2 + 2 (\mu - r) \sigma \frac{W_T - W_t}{T - t} + \sigma^2 \left( \frac{W_T - W_t}{T - t} \right)^2 \), the term (18) further equals
\[
E \left( \int_0^T \left( \frac{\mu - r}{\sigma (T - t)} \right) W_T - W_t \right) + \frac{1}{2} \left( \frac{W_T - W_t}{T - t} \right)^2 dt.
\]

Due to the linearity property of the integral we can consider the two terms of the integrand separately giving (18) in the following form
\[
E \left( \frac{\mu - r}{\sigma} \int_0^T \left( \frac{W_T - W_t}{T - t} \right) dt \right) + \frac{1}{2} E \left( \int_0^T \left( \frac{W_T - W_t}{T - t} \right)^2 dt \right),
\]
in the first term we can see the time integrand of a normally distributed process with zero expected value, so the integrand would also be a normally distributed variable with
zero expectation, so the only remaining part is the second term. For that we may apply Fubini’s theorem as we have a bounded variation term as the integrand. In expectation the squared Wiener-increment is equal to $T - t$, so for (18) we got

$$\frac{1}{2} \int_0^T \frac{1}{T - t} dt = \infty.$$ 

9 Insider information represented as $W_T$ with some noise

Let the insider information be the end point of a $\mathbb{F}$-Brownian motion distorted with a Gaussian white noise

$$G = W_T + a\varepsilon,$$

where $\varepsilon \sim N(0,1)$ is independent of $W$ and $a$ is a constant, therefore $a\varepsilon \sim N(0,a^2)$.

9.1 Finding the density process

Again, it is needed to calculate the density process for the change of measures. The conditional law of $G$ in this case is

$$P [W_T + a\varepsilon \in dx | \mathcal{F}_t].$$

(19)

We can utilize the independence of the Wiener increments and the independence of the noise from the Brownian filtration. As a result the conditional law (19) could be written as

$$P [W_T - W_t + a\varepsilon + W_t \in dx | \mathcal{F}_t],$$

where $W_t$ is $\mathcal{F}_t$-measurable, while both $W_T - W_t$ and $a\varepsilon$ are independent of $\mathcal{F}_t$ and each other. Fixing measurable part as some constant $y$ our expression (19) further equals

$$P [W_T - W_t + a\varepsilon + y \in dx] |_{y=W_t}.$$
We know that both the increment and the noise are normally distributed and independent, therefore their sum is also normally distributed,

\[ A \sim N(m_1, s_1^2) \text{ and } B \sim N(m_2, s_2^2) \Rightarrow A + B \sim N(m_1 + m_2, s_1^2 + s_2^2), \]

adding the constant, in this case the distribution is \( N(y, T - t + a^2) \). Using the density function of the normal distribution, the Equation (19), the conditional law of \( G \) is

\[ \frac{1}{\sqrt{2\pi(T - t + a^2)}} \exp \left\{ -\frac{(x - W_t)^2}{2(T - t + a^2)} \right\} dx. \]

The unconditional law of \( G \) is

\[ P[W_T + a\varepsilon \in dx] = \frac{1}{\sqrt{2\pi(T + a^2)}} \exp \left\{ -\frac{x^2}{2(T + a^2)} \right\} dx. \]

We would need to find the density process \( p_t(\cdot, x) \) that satisfies

\[ P[W_T + a\varepsilon \in dx|\mathcal{F}_t] = p_t(\cdot, x)P[W_T \in dx] \]

or equivalently

\[ \frac{1}{\sqrt{2\pi(T - t + a^2)}} \exp \left\{ -\frac{(x - W_t)^2}{2(T - t + a^2)} \right\} dx = p_t(\cdot, x) \frac{1}{\sqrt{2\pi(T + a^2)}} \exp \left\{ -\frac{x^2}{2(T + a^2)} \right\} dx. \]

The only way this equation holds is by choosing \( p_t(\cdot, x) = \sqrt{\frac{T + a^2}{T - t + a^2}} \exp \left\{ -\frac{(x - W_t)^2}{2(T - t + a^2)} + \frac{x^2}{2(T + a^2)} \right\} \).

### 9.2 Writing the density process in Doléans-exponential form

Please note that the density process for this case is similar to the one, where there was no noise, given by Equation (17). The only difference is that instead of \( T \), here we have \( T + a^2 \). So the usual Doléans-exponential form is simply

\[ p_t(\cdot, x) = \mathcal{E} \left( \int \frac{x - W_s}{T - s + a^2} dW_s \right)_t. \]
9.3 Utility maximization

Applying Girsanov-theorem, every \( \mathbb{F} \)-Brownian motion \( W \) is a drifted Brownian motion under the insider measure

\[
\bar{W}_t = W_t - [W, \mathcal{M}]_t = W_t - \int_0^t \frac{W_T - W_s}{T - s + a^2} ds,
\]

as \( W \) and the \( \mathbb{G} \)-Brownian motion \( \bar{W} \) are both Wiener processes, only differing in their drifts their quadratic covariation is \( ds \).

Again, using that the results are similar to the case where there was no noise, the information drift is

\[
\mu_t (G) = \frac{W_T - W_t}{\sigma (T - t + a^2)}.
\]

And because of the results in Section 3.4 the optimal strategy for the insider would be

\[
\pi_t = \frac{\mu - r + \sigma \frac{W_T - W_t}{T - t + a^2}}{\sigma^2},
\]

which gives that the expected logarithmic utility of an insider is

\[
E \left( \log \tilde{V}_T(x, \pi) \right) = \log x +
\]

\[
+ E \left( \int_0^T \left( \frac{\left( \mu - r + \frac{\sigma \left( W_T - W_t \right)}{\sigma^2 (T - t + a^2)} \right)^2}{2} - \left( \frac{\mu - r + \frac{\sigma \left( W_T - W_t \right)}{T - t + a^2}}{\sigma^2} \right)^2 \right) dt \right) +
\]

\[
+ E \left( \int_0^T \frac{\mu - r + \frac{\sigma \left( W_T - W_t \right)}{T - t + a^2}}{\sigma^2} \sigma d\bar{W}_t \right).
\]

This means that the additional logarithmic utility of an insider is

\[
E \left( \int_0^T \frac{\left( \mu - r + \frac{\sigma \left( W_T - W_t \right)}{T - t + a^2} \right)^2 - (\mu - r)^2}{2\sigma^2} dt \right), \quad (20)
\]
similarly to the previous section, by writing the integrand in a simpler form, we get
\[
E \left( \int_0^T \left( (\mu - r) \frac{W_T - W_t}{\sigma (T - t + a^2)} + \frac{1}{2} \left( \frac{W_T - W_t}{T - t + a^2} \right)^2 \right) dt \right) = \int_0^T E \left( \frac{(W_T - W_t)^2}{(T - t + a^2)^2} \right) dt,
\]
by a similar logic as in the previous section. We know that
\[
E \left( \left( W_T - W_t \right)^2 \right) = T - t,
\]
giving (20) to be
\[
\frac{1}{2} T \int_0^T \frac{T - t}{(T - t + a^2)^2} dt.
\]
The derivative of a logarithmic expression would look similarly
\[
-\frac{\partial}{\partial t} \log (T - t + a^2) = \frac{T - t + a^2}{(T - t + a^2)^2} \]
but we need to apply a corrections of \(-\frac{a^2}{(T - t + a^2)^2}\) to get back this amount. It is easy to see that
\[
-\frac{\partial}{\partial t} \frac{a^2}{T - t + a^2} = \frac{a^2}{(T - t + a^2)^2}.
\]
Then for (20) we get
\[
\frac{1}{2} \left[ -\log (T - t + a^2) - \frac{a^2}{T - t + a^2} \right]_0^T = \frac{1}{2} \left( -1 + \frac{a^2}{T + a^2} + \log \frac{T + a^2}{a^2} \right)
\]
\[
= \frac{1}{2} \left( \log \frac{T + a^2}{a^2} - \frac{T}{T + a^2} \right).
\]

10 Insider information represented as \( G = \int_0^T \varphi(s) dW_s \)

with deterministic \( (\varphi(s))_{s \in [0, T]} \)

In the case where the insider information could be represented as a stochastic integral of a deterministic process \( G = \int_0^T \varphi(s) dW_s \), due to the properties of the Itô-integral, we know that \( G \) is a normally distributed variable with zero expectation and \( \int_0^T \varphi^2(s) ds \) variance. This gives that the results would be the same as in the previous cases, but the conditional expectation is \( \int_0^s \varphi(u) dW_u \) instead of \( W_s \), and the conditional variance is \( \int_0^s \varphi^2(u) du \) instead of \( T - s \) or \( T - s + a^2 \). Therefore we may substitute these values
in the formulas accordingly. The density process becomes

\[ p_t(. , x) = \mathcal{E} \left( \int \frac{x - \int_0^t \varphi(u)\,dW_u}{\int_s^t \varphi^2(u)\,du} \right)_t \]

and applying Girsanov theorem, every \( \mathcal{F} \)-Brownian motion \( W \) is a drifted Brownian motion under the insider measure

\[ \bar{W}_t = W_t - [W, \mathcal{M}]_t = W_t - \int_0^t \int_s^T \varphi(u)\,dW_u - \int_s^T \varphi(u)\,dW_u \]

Using this, the information drift becomes

\[ \mu_t (G) = \frac{\int_s^T \varphi(u)\,dW_u}{\int_s^T \varphi^2(u)\,du} \]

Therefore according to Section 3.4, the optimal strategy of the insider would be

\[ \pi_t = \frac{\mu - r}{\sigma^2} + \frac{\int_s^T \varphi(u)\,dW_u}{\sigma \int_s^T \varphi^2(u)\,du} \]

giving the expected logarithmic utility of the insider to be

\[ E \left( \log \bar{V}_T(x, \pi) \right) = \log x + \frac{T}{2} \left( \mu - r + \sigma \left( \frac{\int_s^T \varphi(u)\,dW_u}{\int_s^T \varphi^2(u)\,du} \right) \right)^2 - \frac{T}{2} \left( \mu - r + \sigma \left( \frac{\int_s^T \varphi(u)\,dW_u}{\int_s^T \varphi^2(u)\,du} \right) \right)^2 ds + \int_0^T \frac{\mu - r + \sigma \left( \frac{\int_s^T \varphi(u)\,dW_u}{\int_s^T \varphi^2(u)\,du} \right)}{\sigma^2} \sigma d\bar{W}_s. \]
And the additional expected logarithmic utility of the insider becomes

\[
E \left( \int_0^T \left( \mu - r + \sigma \left( \frac{\int_0^T \varphi(u) dW_u}{\int_0^T \varphi^2(u) du} \right) \right)^2 \right) - (\mu - r)^2 ds =
\]

\[
E \left( \int_0^T \left( \frac{\int_0^T \varphi(u) dW_u}{\int_0^T \varphi^2(u) du} \right) \right) ds
\]

again as in the first term a Gaussian process \( \int \varphi(u) dW_u \) appears in the numerator with zero expectation. We can apply Fubini’s theorem for the second one, from where

\[
E \left( \int_0^T \left( \frac{\int_0^T \varphi(u) dW_u}{\int_0^T \varphi^2(u) du} \right)^2 \right) \]

\[
= \frac{1}{2} \int_0^T \left( \int_0^T \varphi^2(u) du \right)^{-1} ds.
\]

11 Alternative calculation using Malliavin calculus for \( G = W_T \) case

Let’s go back to the example, where \( G = W_T \). We already saw that the conditional law of \( G \) is normal, therefore may be written as

\[
P \left[ W_T \in dx | \mathcal{F}_t \right] = \frac{1}{\sqrt{2\pi(T-t)}} \exp \left\{ -\frac{(x - W_t)^2}{2(T-t)} \right\} dx,
\]

and to have a compact notation, let \( \frac{1}{\sqrt{2\pi(T-t)}} \exp \left\{ -\frac{(x - W_t)^2}{2(T-t)} \right\} dx \) be \( P_t(dx) \). We want to find a signed measure denoted by \( D_tP_t(dx) \), satisfying Equation (10)

\[
E \left( D_t\psi \left( G \right) | \mathcal{F}_t \right) = \int \psi (x) D_tP_t(dx).
\]
To do so, let’s write the conditional expectation in a different form, using that $D_t \psi (G) = \psi' (G)$ for smooth $\psi$ functions and $G \in D_{2,1}$. Equation (10) becomes

$$E (D_t \psi (G) | \mathcal{F}_t) = E (\psi' (W_T) | \mathcal{F}_t) = E (\psi' (W_t + W_T - W_t) | \mathcal{F}_t).$$

Again using the theorem for the conditional expectation of a measurable and an independent term, (10) further equals to

$$E (\psi' (y + W_T - W_t)) | y=W_t = E \left( \psi' \left( y + \sqrt{T-t} z \right) \right) | y=W_t =

= \int_{-\infty}^{\infty} \psi' \left( y + \sqrt{T-t} z \right) \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{z^2}{2} \right\} dz,$$

if $z$ is a standard normal variable, because $W_T - W_t \sim N(0, T-t)$. Keeping in mind that our main goal was to have the integral of $\psi(x)$ on the right hand side, using the partial integration formula

$$\int fg' = fg - \int f'g, \text{ choosing } f = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{z^2}{2} \right\} \text{ and } g' = \psi' \left( y + \sqrt{T-t} z \right),$$

$f' = -zf$ and $g = \frac{\psi(y+\sqrt{T-t}z)}{\sqrt{T-t}}$. Therefore the Equation (10) further equals

$$\frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{z^2}{2} \right\} \cdot \psi \left( y + \sqrt{T-t} z \right) \sqrt{T-t} + \int_{-\infty}^{\infty} \psi \left( y + \sqrt{T-t} z \right) \cdot \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{z^2}{2} \right\} dz.$$

From which, the first term is zero, as it contains the probability of a standard normal variable being a single value $z$. The standard normal distribution is continuous and the probability of any point would be zero. Writing the integral back to the expectation form, we get that (10) becomes

$$E \left( z \cdot \psi \left( y + \sqrt{T-t} z \right) \right) | y=W_t = E \left( \psi (W_T) \frac{W_T - W_t}{T-t} \right).$$

This means that the Radon-Nikodym derivative between the sought and the original measure must be

$$\frac{D_t P_t(dx)}{P_t(dx)} = \frac{W_T - W_t}{T-t} (dx).$$
to get to the desired form. It also gives that

$$\tilde{W}_t = W_t - \int_0^t \frac{W_T - W_s}{T - s} ds,$$

just like in Section 8. From this point on, all the further steps are the same.
Chapter VI

Summary

In this thesis I dealt with the topic of modelling insider information. I considered a regular investor and an insider. They were both well-informed, but the insider also knew an additional information that was not available for other market participants. I assumed a market with price processes driven by a local martingale and its quadratic variation. Using the Jacod condition, it was possible to define an insider measure, under which it was easier to deal with the insider information.

Throughout my thesis, I assumed that both type of investors have the logarithm of terminal discounted wealth as a utility function, which is a common assumption. This type of utility function cannot reflect an investor’s utility perfectly, but it is still quite accurate, easy to use and there are no better alternatives currently. To maximize the logarithmic utility I had to define the discounted wealth process, which was determined by the strategy $\pi$, giving the ratio of the total invested wealth kept in each products. Based on the market dynamics and the filtrations of each investor, the optimal strategy could be calculated. The additional logarithmic utility of the insider was simply the difference between the expected logarithmic utility of the insider and the regular investor.

There are cases where Jacod’s condition does not hold, but are still relevant in practice. Using the Clark-Ocone formula and Malliavin calculus gives an alternative way to treat the enlarged filtration. The basics of this theory was also studied in this thesis.

At the end, I applied the theories above to a Black-Scholes market. I made different assumptions on the insider information and calculated the additional logarithmic utility of the insider in these special cases, showing simple examples on how to interpret the Jacod framework and the approach using Malliavin calculus.

There are many interesting questions related to the topic of modelling the insider trading, which could be further investigated. For example, given the techniques presented in this paper, there are papers focusing on the arbitrage possibilities. Although the techniques shown in this paper enables us to calculate the additional logarithmic utility of the insider, it may be possible that there are other ways to apply the enlargement
of filtrations technique making weaker assumptions only. This way we might deal with
different types of insider information, too.

If we would find a utility function that better describes the preferences of the investors,
we could modify our assumptions on the utility function of the investors. This would
lead us to different optimization problems, which may require different theorems to
solve.

Stepping out of the framework presented here, it would be also possible to deal with the
problem of insider trading without using the enlargement of filtrations. For example, if
we would assume that the actions the insider and the regular trader make also has an
influence on the asset prices, the assumptions made on the dynamics of the financial
instruments would not hold. We would need to model both the influences the investors
make and their strategy would need to be modified to react to these changes.

As a conclusion, the topic of insider trading still holds a lot of questions to investigate.
Both Jacod’s condition and Malliavin calculus provided a framework, under which
we could further examine the additional utility of the insider using enlargement of
filtrations approach.
References


