Rough Volatility in Foreign Exchange Markets

Forecasting the realized volatility of currency markets with the Rough Fractional Stochastic Volatility Model

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Abstract

In recent years the application of fractional Brownian motion in financial modelling has become widespread both in theoretical finance and financial econometrics. In their seminal paper Gatheral et al (2014) have shown that, their fractional Brownian motion-based model performs better, than standard econometric models at predicting realized volatility of different equity indexes. In this study we will analyze the evolution of the volatility of different liquid and illiquid foreign exchange rates. Our hypothesis is that FX volatilities exhibit the same smoothness feature as equity volatilities. Despite the different characteristics of the volatility on currency markets we expect the Rough Fractional Volatility model to perform well in predicting the realized variances. In order to test our hypothesis we used high-frequency datasets for 6 currencies. Our results are encouraging, because our model proved to be a superior forecasting tool of realized variance compared to other econometric models.

JEL-code: G17

Keywords: foreign exchange, volatility forecast, fractional Brownian motion
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I. Introduction

In traditional financial modelling, the price of underlying assets are assumed to evolve according to a geometric Brownian motion. This means that the driving force behind the evolution of the volatility of the price process is a Brownian motion. Gatheral et al (2014) introduced a model called Rough Fractional Stochastic Volatility model, where the evolution of the volatility is determined by a fractional Brownian motion. The fractional Brownian motion can be considered as a generalization of the Brownian motion. Depending on the parametrization, it can reproduce Brownian motion, long memory process or so-called rough processes. They have analyzed high-frequency data from various equities and equity indices, and came to the conclusion that in a sense these processes are “rougher”, than what would be implied by a standard Brownian motion. The economic intuition behind their idea was the emergence of high-frequency trading. They argue that these frequent trades “drag” the price process so much that their paths become inherently different from the paths of a geometric Brownian motion. In their seminal paper the authors have also analyzed how their model performs in forecasting the volatility. They have showed that their model performs better than benchmark time-series models at forecasting the realized volatility. Since then, many other authors have analyzed the roughness of financial processes, both in theoretical finance like option pricing, for example Bayer et al (2015), and econometric modelling, like Bennedsen et al (2016).

In our study we will analyze 5-minute, high-frequency data for different liquid and illiquid currencies, and forecast the evolution of the volatility of these currencies. Our research question is whether currency markets exhibit the same volatility smoothness characteristics as equity markets or not. This question is relevant because the trading on foreign exchange markets is quite different from that on stock markets, and the characteristics of the implied volatility surface is generally different from what we can observe for equity markets. We will also examine whether there are differences in this sense between liquid (EURUSD, USDJPY and EURGBP in our dataset) and less liquid currencies (EURPLN, EURRON and EURHUF in our dataset). Our hypothesis is that the Rough Fractional Stochastic Volatility model will perform better than traditional econometric models at forecasting the realized volatility of both liquid and less liquid currencies.

This study is going to start by introducing the most important stylized facts about volatility in the next chapter. These are features that appear regularly in scientific papers, when the authors are
trying to investigate the empirical characteristics of volatility. Then, we are going to introduce the concept of implied volatility. Here, we will also show the reasons why the implied volatility has led to many generalizations of the famous Black-Scholes model. However, we will also highlight the fact that these industry-standard models are unable to capture the term structure of the implied volatility surface. After that, we will introduce the models that we use to forecast the realized variance of our data. Most importantly here we will introduce the Rough Fractional Stochastic Volatility model (RFSV) in detail, which is the main focus of this study. Next, we will show the results of our models, and show that the RFSV model outperforms other important volatility forecasting models. Then, we draw the conclusions from our study.

II. Stylized facts about volatility

In this section we are going to present a methodology to accurately measure the underlying volatility of a financial time series. After that we will give an overview of the most notable stylized facts about the volatility of time series that have been documented in the past decades.

1. Calculating the realized variance measure

To be able to study the volatility of a time series, we are going to need an unbiased measurement of the underlying volatility. For this, let’s suppose that the logarithmic price of an asset follows a diffusion process. Then the return of the asset can be formulated as follows. (Andersen et al, 2001; Patton-Sheppard, 2015 pp. 684)

\[ r(t) = \int_{t-\tau}^{t} \mu(s)ds + \int_{t-\tau}^{t} \sigma(s)dW(s) \]  

(1)

Here \( \mu \) and \( \sigma \) parameters are the drift and deviation components of the process, \( W \) is a standard Wiener process, and \( \tau \) is the size of the time increment. In this case the quadratic variation of the process equals to the so called integrated volatility (IV), which plays a central role in many option pricing models such as the one presented by Hull and White in 1987. (Andersen et al, 2001; Patton-Sheppard, 2015 pp. 684)

\[ IV(t) = \int_{t-\tau}^{t} \sigma^2(s)dW(s) \]  

(2)
The problem is, that the $\sigma$ term in the previous equation is a latent value, therefore we can’t measure it exactly. To solve this problem Andersen et al (1999) show that if we sum high frequency returns during a time period $[t, t + \tau]$, then we get an unbiased measure of integrated volatility, called the realized variance (RV). (Patton-Sheppard, 2015 pp. 684)

$$RV_t = \sum_{k=1}^{n} r_k^2$$ (3)

In our case the length of time period $[t, t + \tau]$ is one day, and we split our time period into $n$ intervals of the same length. This way $r_k^2$ means the squared log-return of interval $k$ (e.g. if $n = 12$ then we sum the squared hourly returns over day $t$). This measure converges in probability to the integrated volatility as $n$ approaches infinity (in other words the intervals between returns become infinitesimally small). But this is only true, if the returns are governed by the stochastic process formulated in equation 1. In the real world at really high frequencies, microstructure effects - such as the bid-ask bounce effect - make the presented RV measure biased. Therefore it is a common practice to use only 5 minute data to calculate the RV measure. Of course the optimal length of the time intervals can differ according to the liquidity of the asset. But this method nevertheless gives us an estimate that is not biased by microstructure effects and at the same time accurate enough for our purposes. Later in this paper we are going to use this 5-minute methodology as well when fitting our models. (Andersen et al, 2001; Patton-Sheppard, 2015 pp. 684)

2. **The distribution of log-volatility**

A well-known stylized fact is that the distribution of log-volatility is approximately Gaussian. This empirical phenomenon is analyzed in for example Andersen and Bollerslev (1997) and in Andersen et al (2001). However - as we will discuss later in more detail - Gatheral et al (2014) model the underlying volatility of the time series as a stochastic process, the increments of which are also normally distributed (since the difference of two normally distributed variables is also normal).

We have also illustrated this phenomenon in two major equity indices that Gatheral et al (2014) used in their paper, the S&P500 and the Dow Jones Industrial Index. For these histograms we have
used data from the Oxford-Man database, and the realized volatility was calculated for 5-minute data in the same way as we have already introduced. (Gerd et al, 2009)

Figure 1 - The distribution of the increments of log-volatility
Own figure, based on Gerd et al database (2009)

We can see that, there is only little deviation from the normal distribution in both examples. (Note that although the Figures are not standardized, the distribution of the increments of log-volatility closely resembles that of a standard-normal variable.) Now, we will examine this phenomenon in our foreign exchange (FX) data.

Figure 2 – The distribution of delta-log-volatilities across FX rates
Own figure, based on Bloomberg database (2017)
We can see that the distributions of the increments of the log-volatility are generally more leptokurtic, than the normal distribution. It is also interesting to see that this feature is stronger in the case of the less liquid currencies. From this, we could conclude that the reason for the difference between the distribution of the increments of equity and exchange rate log-voltalities lies in the fact that currency markets are traditionally less liquid than equity markets.

3. Clustering and long memory

It is a well-known fact that the returns of financial assets contain no autocorrelation. This is not so surprising considering the well known efficient market hypothesis of Fama (1970). The hypothesis states, that if markets are efficient (meaning that the price of assets reflect all information available) than we cannot make assumptions about next day’s return using historical return data. Therefore there is no autocorrelation between lagged values of daily returns. (Ding et al, 1993)

However, this is not the case if we look at absolute returns, which is an imperfect proxy for the underlying volatility of the price process. Examining the absolute returns of a typical asset (e.g. a stock) we can see that there is statistically significant autocorrelation between its lagged values. This phenomenon is the clustering of volatility. It means that the underlying volatility of a financial time series has significant autocorrelation, and time periods with high (low) volatility tend to be followed by other high (low) volatility periods. (Ding et al, 1993)

Not only that, but this autocorrelation is significant even after several lags. If returns would follow a process similar to the one modelled by generalized autoregressive conditional heteroscedasticity models (GARCH for short), we would expect the autocorrelation function to have an exponential rate of decay. In contrast to this expectation, Ding, Granger and Engle (1993) show that the autocorrelation function of absolute returns decays hyperbolically rather than exponentially. Following Ding et al (1993) consider the following measure.

\[ |r_t|^d = |\ln(p_t) - \ln(p_{t-1})|^d \]  

(4)

Here \( p \) denotes the price of the asset and \( d \) is a positive number. The authors show, that the aforementioned hyperbolical rate of decay is true even for \( d \) values greater than one. (They also point out that autocorrelation is greatest when \( d \) parameter is just above one.) This is what we call the long memory property of a time series. Mathematically a time series follows a long memory
process, if the following equation is true for its autocorrelation function (Maekawa et al, 2015 pp. 2).

\[
\sum_{k=1}^{\infty} |\rho(k)| = \infty
\]

Whether or not financial time series truly follow a long memory process, or just have specific properties that resemble long memory, has been a controversial question in the past few years. Many statistical and mathematical models have been proposed to imitate this property of returns, for example Corsi’s (2009) Heterogeneous Autoregressive (HAR) model, the fractional autoregressive-moving-average model (ARFIMA), or the Rough Fractional Stochastic Volatility model (RFSV), all of which we are going to talk about in detail in later sections of this paper. (Ding et al, 1993)

4. Leverage effect and volatility feedback

The relationship between returns and volatility has been studied by many authors as well. According to Brooks (2008) the leverage effect means that the volatility changes in a different way, depending on whether the price of the underlying asset goes up or down. Empirically, the volatility tends to increase more after negative returns than after positive ones. Bollerslev et al (2006) point out that this could be explained by the investors’ behaviour. When there is a sudden negative shock on the market, investors quickly react in fear and drive up volatility.

Volatility feedback means exactly the opposite. It means that news about increasing volatility causes future returns to be negative. Bollerslev et al (2006) study both leverage and volatility feedback effects, and conclude that while the former can be empirically confirmed, there is no evidence for volatility feedback in real-world data.

Historically this asymmetric behaviour has led to the development of many advanced GARCH-type models, like the Exponential-GARCH or the GJR-GARCH models, and even in some versions of the HAR models, about which we are going to talk about in more detail later in this paper.
5. Smoothness of the volatility process

In this section, we are going to estimate the smoothness of the volatility processes of our currency data. First, we have to define what is meant by the smoothness of a process. Following Rosenbaum (2011), we will assume for now that the real volatility process is observable at N discrete time points with \( \Delta \) distance between each step, and define

\[
m(q, \Delta) = \sum_{i=0}^{N} |\log(\sigma_{i\Delta}) - \log(\sigma_{(i-1)\Delta})|^q.
\]

Rosenbaum (2011) assumes that for \( s_q > 0 \) and \( b_q > 0 \), if \( \Delta \) goes to zero then

\[
N^{qs_q} m(q, \Delta) \to b_q.
\]

We will refer to \( s_q \) as the smoothness of the volatility process. This \( s_q \) parameter will be important for us when we define the Rough Fractional Stochastic Volatility model, because it will determine whether the log-volatility exhibits the long memory behaviour or not. The volatility is of course not observable, so we will use RV as a proxy variable.

Next we are going to estimate the smoothness parameter with linear regression between the log\((m(q,\Delta))\) and the log\((\Delta)\), which comes from calculating the \( m(q, \Delta) \) for a given \( q \) and \( \Delta \). The result above from Rosenbaum (2011) shows, that the slope of this regression line is exactly the \( s_q \) smoothness parameter. Bennedsen et al (2016) and Gatheral et al (2014) showed that for equities this parameters are definitely lower, than 0.5 (we will show later, why the value 0.5 is an important boundary). The charts below shows the results of the regression models.
We can see that the estimates of the smoothness parameters are very low. The table below contains the exact parameter estimates.

<table>
<thead>
<tr>
<th>FX rate</th>
<th>Smoothness</th>
</tr>
</thead>
<tbody>
<tr>
<td>EURUSD</td>
<td>0.0918</td>
</tr>
<tr>
<td>EURGBP</td>
<td>0.1236</td>
</tr>
<tr>
<td>USDJPY</td>
<td>0.1056</td>
</tr>
<tr>
<td>EURHUF</td>
<td>0.0901</td>
</tr>
<tr>
<td>EURPLN</td>
<td>0.1418</td>
</tr>
<tr>
<td>EURRON</td>
<td>0.0208</td>
</tr>
</tbody>
</table>

We can see that all of the estimates besides the parameter of EURRON are around 0.1, so they are significantly lower, than the 0.5 boundary. The parameter of EURRON is very low, the reason for this is not clear, we can see that the fit of the regression lines is significantly worse, than the fit of the other regression lines. (Later in this paper we will also point out that the EURRON exchange rate is an outlier in other aspects as well.)
6. Volatility smiles

The option pricing model introduced by Black and Scholes (1973) uses several variables to determine the value of a European option. One of the variables is the implied volatility which is the investors’ *ex ante* expectation about future volatility, therefore it cannot be examined. However we can observe all the other variables (prompt price of the underlying asset, strike price, risk free return and maturity) as well as the price of call and put options from market data. Therefore we can use several methods to calculate the implied volatility. To do this we can use the Black-Scholes (B-S) model, or other model-free methods like the one introduced by Carr and Madan (1998).

Whichever methodology we use to calculate the implied volatilities, one would expect the results to be the same across all strike prices if the options are linked to the same underlying asset. However a well-known anomaly on option markets is the so-called volatility smile. The term smile is a reference to the typical shape of the implied volatility curve if we plot it against the strike prices of the options. Investors who buy deeply out-the-money (OTM) or in-the-money (ITM) options tend to expect higher future volatilities for the same underlying asset than others buying at-the-money (ATM) options. (Hull, 2015)

This phenomenon could have a couple of possible explanations. One of them is that there is inefficiency on the option market and a special investing strategy could take advantage of this resulting significant positive returns. However Ederington and Guan (2002) point out that although such a strategy might work in theory, the transaction costs faced during its execution would make the potential profit insignificant. Another explanation could be that the assumption of the B-S model that the price of the underlying asset follows a geometric Brownian motion is false. This would mean that the volatility smiles are because tail-events have greater probabilities of happening than the B-S model would suggest. Ederington and Guan also point out that many studies have tried to create option pricing models with time-varying volatility that better mimic real world data, but they haven’t been able to fully account for the smiles observed on the market. (Ederington-Guan, 2002; Hull, 2015)

One last explanation for the volatility smile that should me mentioned is hypothesized by Ederington and Guan (2002). The authors claim that the volatility smile could come from hedging pressure by investors, who wish to protect their investments against possible extreme events. This
theory could also explain the main difference between volatility smiles on stock and FX markets. The volatility smiles on equity markets tend to be skewed, meaning that the implied volatility of OTM puts options are larger than that of OTM calls. Investors are only afraid of negative crashes on the market but not of positive jumps. Therefore they drive up the prices of OTM put options to secure their portfolios (and the resulting anomaly cannot be hedged away due to reasons mentioned before). On the other hand is the FX volatility smile, which is more symmetrical than the equity options’ smile. This is logical considering the hypothesis of Ederington and Guan. In the case of foreign exchanges the distribution of the returns depends on which side of the cross we’re looking at. Not only that but also great positive changes in a FX price could be as bad news for investors as negative changes of the same magnitude. A sudden appreciation of a country’s currency could also cause harm for the country’s economy as the depreciation of it. Therefore it is not surprising that the shape of volatility smiles differ between equity and FX markets. These observations also support our motivation to further investigate the performance of volatility models on FX markets. (Hull, 2015)

7. Term structure of implied volatility

One important stylized fact about the volatility smile is that its level and orientation changes over time in equity markets. For currency options, this is not trivial, because we have illustrated in the previous section that the implied volatility surface is more symmetric. Besides that, the skew does not decay over time. Gatheral (2006) defined the term structure of volatility skew for at-the-money options as

$$\varphi(\tau) = \left. \frac{\partial}{\partial k} \sigma_{BS}(k, \tau) \right|_{k=0},$$

where $k$ indicates the moneyness of a given option contract. This formula can be approximated by a function that decays as a power-law function, which is shown in the chart below.
We can see from the chart that the volatility skew is decreasing as a power-law function over time, which is the consequence of the features of the volatility surface from the previous section. As Gatheral et al (2014) point out one of the attractive features of the RFSV model introduces by them - and also examined by us in this paper – is that unlike other volatility models it is able to replicate this kind of term structure when the model is fitted to empirical volatility surfaces. In the case of foreign exchanges the shape of the skew term structure however is quite different. This fact further supports our research question, whether or not it the RFSV model will be able to be a superior predictor of FX variances as well as equity variances.

In the next section we will introduce some generalizations of the Black-Scholes model, which tried to account for the fact that the volatility is not constant. However, we will also show how those models are unable to reproduce this dynamics of the implied volatility surface.

III. Generalizing the Black-Scholes model

Many generalized models have evolved to account for these properties of the implied volatility. These models can be considered as a generalization of the classical Black-Scholes model (Black-Scholes, 1973). In this section, we will introduce some examples of these models briefly. The introduction will mainly focus on the intuition of the models, and the problems one has to face, while trying to reproduce the term structure of volatility introduced in the previous section. This section is important, because it gives the intuition behind the Rough Fractional Stochastic Volatility, which is the main focus of our study.
1. Local volatility models

Historically, the first generalization was the family of the local volatility models, where the volatility is no longer a constant, rather it will be a function depending on time and the stock price. So, the classical geometric Brownian motion is changed to

\[ dS_t = \mu S_t dt + \sigma(t, S_t) S_t dW_t \]  \hspace{1cm} (9)

In theory, if we choose \( \sigma(t, S_t) \) function well then the volatility implied by equation 9 equals to the actual implied volatility. The calibration of this function is achieved by actually calculating the function from option data available at the market. Local volatility models are discussed in great details in Dupire (1994) and Lipton (2001).

There are many drawbacks of using these type of models. Here, we would like to list some of these drawbacks, the reader can go into details about these in for example Hagan et al (2002).

- Local volatility models are highly unstable
- It is hard to interpolate between observed data, because you have to guarantee that the no-arbitrage condition is satisfied.
- These models create false hedging strategies
- Local volatility models create highly unlike dynamics of the implied volatility. This is consequence of fitting to given dates.

The most important problem for our model is the fact that local volatility models are unable to reproduce the term structure of implied volatility introduced above. These types of problems has led to the development of stochastic volatility models, which will be introduced in the next section.

2. Stochastic volatility models

In order to account for these problems, researchers have tried to generalize the definition of the volatility even more. Thus, they defined volatility as a stochastic process, and developed two-factor model, a prime example of which is the Heston model defined below

\[ dS_t = \mu S_t dt + \sqrt{\nu_t} S_t dW_t^S \]  \hspace{1cm} (10)
\[ dv_t = \kappa(\theta - v_t) + \xi \sqrt{v_t} dW^v_t \]
\[ dW^S_t dW^v_t = \rho dt \]

This model was formulated by Heston (1993). The volatility process is defined as a mean-reverting process to \( \theta \) as the long-term average. Stochastic volatility models have become industry-standards, however Fukasawa (2011) showed that the most popular stochastic volatility models, for example the Hull and White, Heston and SABR models, are unable to reproduce the term structure of volatility. Typically these models generate constant term structure for small values of \( \tau \), and then it is decreasing as an exponential function. Fukasawa (2011) also showed that the basic stochastic volatility model can be re-formulated with a fractional Brownian motion in a way to exhibit the power-law behaviour of the term structure of volatility. He showed that in order to model this phenomenon properly we need a process that is “rough”, which means that its Hurst parameter is relatively low, this concept will be introduced later in this study. This result is highly encouraging for our Rough Fractional Stochastic Volatility model, because it relies heavily on the fractional Brownian motion process.

This chapter showed that some of the most important quantitative finance models are unable to produce the term structure of the implied volatility. The local volatility models in most cases create completely false dynamics, and the stochastic volatility models cannot reproduce the power-law behaviour of the volatility term structure. Gatheral et al (2014) showed that the model we use primarily in this study performs better in these terms.

**IV. Forecasting models**

In this section we are going to briefly discuss some of the more important models to consider if we are to forecast a time series’ volatility.

1. **AR model**

Probably the most famous model to describe a time series’ underlying volatility the autoregressive conditional heteroscedasticity (ARCH) model proposed by Engle in his 1982 paper, which has led to a series of new, improved volatility models such as a generalized ARCH (GARCH) model of Bollerslev (1986). The main idea of these models is that the underlying volatility of financial time series follow an autoregressive process. This explains the clustering of volatility that we described
earlier: periods with high volatility tend to follow high volatility periods and *vica versa*. This can be formulated with the following two equations.

\[ r_t = \mu + \varepsilon_t \quad \text{where} \quad \varepsilon_t \sim N(0, \sigma_t^2) \]

\[ \sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 \]

The first equation of 14 is the so-called mean equation (modelling the returns as a stationary project) where \( \varepsilon_t \) is a normally distributed random variable with a zero mean and \( \sigma_t^2 \) variance. The second equation is the variance equation. We can see that it models the latent \( \sigma_t^2 \) variable with the lagged values of \( \varepsilon \) squared. We can rearrange this equation to highlight the autoregressive property of it by adding \( \varepsilon_t^2 \) to and subtracting \( \sigma_t^2 \) from both sides. This way we get Equation 15. (Engle, 1982)

\[ \varepsilon_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + u_t \] (15)

This way we can see that equations 14 assume that \( \varepsilon_t^2 \) follows a first-order autoregressive process (AR), where \( u_t \) is the error term of the model and is equal to \( \varepsilon_t^2 - \sigma_t^2 \). Because \( \varepsilon_t \) is the demeaned return of time period \( t \), \( \varepsilon_t^2 \) can be seen as an imperfect proxy for the time period’s volatility. Therefore if we substitute \( \varepsilon_t^2 \) with the aforementioned RV measure (which is a much better measurement of underlying volatility) we might be able to use a first-order AR model to forecast future volatility. Of course there are a lot of more sophisticated models that can produce superior predictions compared to the AR model. But we wish to use this model as a reference point, to which we can compare the other models’ performance. (Engle, 1982)

2. ARFIMA model

The autoregressive fractionally integrated moving average (ARFIMA) model is one of the most common econometric models for long memory time series. ARFIMA models assume that the time series are fractionally integrated (Maekawa et al, 2015).

\[ \Phi(L)(1 - L)^d y_t = \Theta(L)\varepsilon_t \]

\[ \Phi(L) = 1 - \Phi_1 L - \Phi_2 L^2 - \cdots - \Phi_p L^p \] (16)
\[ \Theta(L) = 1 - \theta_1 L - \theta_2 L^2 - \cdots - \theta_q L^q \]

The \( \Phi \) and \( \Theta \) terms of the equations are the same as in the traditional ARIMA\((p,d,q)\) model, where they represent the autoregressive and moving average coefficients with \( L \) being the lag operator. The difference between ARIMA and ARFIMA models is the fractional difference operator \( (1 - L)^d \) which can also be expressed by Binomial series. (Maekawa et al, 2015 pp. 2)

\[
(1 - L)^d = 1 + \sum_{k=1}^{\infty} \frac{d(d-1) \cdots (d-k+1)}{k!} (-L)^k
\]  \hspace{1cm} (17)

With the Binomial series we can define differentiation for time series, and for non-integer \( d \) values. (Notice that for positive integer values of \( d \) we get back the classical ARIMA\((p,d,q)\) model.) This guarantees that the model has a long memory property, and that the autocorrelation function will decay hyperbolically rather than exponentially. (Maekawa et al, 2015 pp. 2)

In practice the ARFIMA model tends to produce superior forecasts of volatility than short memory models, therefore later on in this paper we are going to fit this model to our data as well. The main criticism this model gets is that it is not intuitive and its results are hard to economically interpret. Next we will look at a model introduced by Corsi, that aims to mimic the long memory properties of time series, performs just about as good as ARFIMA, and is easier to interpret. (Corsi, 2009)

3. HAR model

In his groundbreaking 2009 paper Corsi introduced a new econometric model to forecast the volatility of a time series, called the Heterogeneous Autoregressive Realized Variance model (HAR-RV). The central idea in his paper is similar to the ideas discussed so far: simulations from classical volatility models (like GARCH) cannot reproduce some of the stylized facts that we have talked about earlier in this paper, such as long memory. As mentioned in the previous section the ARFIMA model is able to simulate time series with long memory, but has the drawback of lacking clear economic intuition behind it. With his new model the author wishes to solve these two problems at the same time. (Corsi, 2009)

The main idea behind the model is that there are heterogeneous investors on the market with different trading horizons. On one hand there are institutional investors, who make investment
choices less frequently but invest greater amounts of wealth. On the other hand are small investors who make decisions on a daily basis. These different type of actors are influenced by and cause different volatilities on different time horizons. The individual investor who makes decisions daily will be interested in yesterday’s, last week’s and even the previous month’s volatility. However, the realized volatility of yesterday will have little effect on bigger investors’ decisions who only adjust their portfolios on a monthly basis. (Corsi, 2009)

To address this heterogeneity of actors on the market, Corsi proposes the following model that incorporates realized variance measures of different time horizons. (Corsi, 2009 pp. 181)

\[
RV_{t+1d} = c + \beta(d)RV_t^{(d)} + \beta(d)RV_t^{(w)} + \beta(m)RV_t^{(m)} + \omega_{t+1d}
\]  

(18)

The upper indices are indicating whether the realized variance measures represent daily, weekly or monthly volatilities. A weekly and monthly RV measures of time \( t + 1d \) were constructed as a simple sum of the daily realized variances of the past 5 and 22 working days. As can be seen from equation 18 the model assumes a linear relationship between daily variances and volatilities of all three horizons. A further advantage of this model is that it can be estimated using a simple OLS regression. (Corsi, 2009)

HAR has proven to be a highly accurate predictor of daily realized variance compared to other models and at the same time is very easy to interpret. Since the proposal of this model there has been several papers that modify equation 18 to further improve its performance. For example Patton and Sheppard (2015) introduced signed jump variation measures and bipower variation measures to their version of HAR, and showed that by including these extra variables the predictions can be further improved, making HAR a state of the art model for volatility forecasting.

V. The Rough Fractional Stochastic Volatility model

In this section we will introduce a relatively new model, called the Rough Fractional Stochastic Volatility model. Based on the empirical results from the previous chapters Gatheral et al (2014) proposed a simple, but powerful way to model the dynamics of the volatility process.
1. Mathematical background

In this part, we are going to introduce the mathematical building-block of the RFSV model, the Fractional Brownian Motion. We will start with defining a simple Brownian motion.

**Brownian motion** *(Definition based on Shreve (2013 pp. 94)):* $B_t$ is a Brownian motion if

1. $B_0 = 0$
2. The trajectories of $B_t$ are continuous
3. $B_t$ has independent increments
4. $B_t - B_s = N(0, t - s), \forall s < t,$

where $N(0, t - s)$ denotes normal distribution with 0 expected value, and $t - s$ variance

The Brownian motion is an essential part of modern financial mathematics. For instance the ground-breaking work of Black, Scholes (1973) is based on the geometric Brownian motion, the basis of which is the simple Brownian motion. One can read more about these topics in for example Shreve (2013).

The most important part of our model of the increments of log-volatility is the fractional Brownian motion, which was defined by Mandelbrot and van Ness (1968). This process is a generalization of the simple Brownian motion,

**Fractional Brownian motion** *(Definition based on Mandelbrot and van Ness (1968 pp. 256)):*

*Fractional Brownian Motion is Gaussian process with zero expected value, and an autocovariance function in the following form

$$\text{Cov}(B^H_t, B^H_s) = 0.5 \times \left[ |t|^{2H} + |s|^{2H} - |t - s|^{2H} \right]$$

(19)

In the definition $H$ is called the Hurst-parameter or Hurst index, which takes values from the $(0, 1)$ interval. This parameter determines the autocovariance structure of the fractional Brownian motion, because if

- $H = \frac{1}{2}$, then we get the standard Brownian motion
- $H > \frac{1}{2}$, then the process exhibits the long – memory feature
• $H < \frac{1}{2}$, then the increments of the process are negatively correlated

The fBM is said to possess rough paths, if the last condition is satisfied, because in this way the smoothness parameter of the process is lower, than the smoothness of the standard Brownian motion. This process has some interesting properties. First, the process is self-similar.

$$B_t^H \sim |c|^H B_t^H$$

This means that the left and the right side are equivalent in distribution. This can also be considered as a fractal property. The fractal behaviour of the log-volatility has been studied extensively in for example by Bacry and Muzy (2003). This is a significant similarity between the fractional Brownian motion and the log-volatility process. The second interesting property is the regularity. The trajectories of this process are not differentiable almost everywhere, but they satisfy the H"older-continuity with any order strictly less, than the Hurst parameter (Mandelbrot and van Ness, 1968).

2. Model specification

Gatheral et al (2014) and Bennedsen et al (2016) showed that for various type of equity assets, the increments of log-volatility follow a process with a constant and quite small smoothness parameter. They have also showed that the distribution of the increments is approximately Gaussian.

From the definition of the fractional Brownian motion (fBM from here) and these empirical properties, it would be intuitive to propose a statistical model of the log-volatility based on an fBM.

$$\log \sigma_{t+\Delta} - \log \sigma_t = \gamma (W_{t+\Delta}^H - W_t^H)$$

Here $\Delta$ denotes the time step we take between our observations and $W_t^H$ is an fBM with $H$ as the Hurst parameter, which will be modelled by the smoothness parameter, defined in the stylized facts section. We have shown before that the empirical smoothness parameter is significantly lower than 0.5, which means that the log-volatility is not a long-memory process.
It would be reasonable, from the arguments above, to assume that this model describes the dynamics of the log-volatility well. However, there is a huge problem involving this model, which can be easily seen from the equation below.

\[ \sigma_t = \sigma \exp\{\gamma W_t^H\} \]  

It is quite clear that this model is not stationary. In econometrics, most of the times we want to use stationary models in order to create reasonable long-term dynamics. In order to satisfy the stationarity condition in their model Gatheral et al (2014) assumed that the increments of the log-volatility follows a fractional Ornstein-Uhlenbeck process (fOU from here). The fOU process is a mean-reverting process, so, in order to re-create the empirical properties of the increments of log-volatility, we will use a very long mean-reversion time.

\[ dX_t = \gamma dW_t^H - \alpha (X_t - m) dt \]  

The explicit solution of the fOU stochastic differential equation is given by

\[ X_t = \gamma \int_{-\infty}^{t} \exp\{-\alpha (t - s)\} \, dW_s^H + m. \]  

Concluding, the final specification of the Rough Fractional Stochastic Volatility model is given by the next equation, where \( X_t \) is the solution of a fOU stochastic differential equation.

\[ \sigma_t = \exp\{X_t\} \]
Here, we would like to emphasize that, it can be seen from the definition of the fBM process that the RFSV does not exhibit the long memory property, so the estimated volatility process will not be a long memory process. This feature contradicts with the well-established long-memory stylized property studied by for example Andersen and Bollerslev (1997). However, Gatheral et al (2014) showed that by using traditional statistical procedures, which aim to identify long memory, on time series generated by the RFSV model, then we would easily deduce that these time series exhibit the long memory feature. The authors claim that this spurious long memory is the result of the strong modelling assumptions of the statistical tests.

3. Forecasting with the RFSV model
In this section we will propose the theoretical foundations of the last part of our work. We will introduce how to forecast the volatility process with the previous model. Nuzman and Poor (2000) calculated the following formula, which is essential to derive our forecasting formula

\[
E[W_{t+\Delta}^H | I_t] = \frac{\cos(H\pi)}{\pi} \int_{-\infty}^{t} \frac{W_s^H}{(t-s+\Delta)(t-s)^{H+0.5}} ds \quad (26)
\]

This formula shows the conditional expected value of the \(\Delta\) period ahead value of the fBM process. We have formulated the RFSV model according to equation (22), and we have argued that at any reasonable, bounded time-scale the fOU process can be considered as the fBM process. These arguments lead to the following forecasting formula from Gatheral et al (2014).

\[
E[log\sigma_{t+\Delta}^2 | I_t] = \frac{\cos(H\pi)}{\pi} \int_{-\infty}^{t} \frac{log\sigma_{t+\Delta}^2}{(t-s+\Delta)(t-s)^{H+0.5}} ds \quad (27)
\]

We will use this formula with a little change, because we are going to perform rolling-window forecast. The only change we will make is naturally in the bounds of the integral.

VI. Data

1. Data Description
As we have mentioned earlier, Gatheral et al (2014) point out that the RFSV model presented in their paper has the advantage of fitting the volatility surface better than any other volatility model. (Note that they examined the same forecasting models that we do in this paper: AR, ARFIMA and
HAR.) They also note that the superior performance of the RFSV model could attributed to the fact that it successfully models microstructural dynamics that are typical of equity markets. For these reasons we have carried out our own empirical research, using foreign exchange data to test whether the RFSV performs superior only in the case of equity time series or in the case of foreign exchanges as well. For this we have selected six foreign exchange rates. Three of them that have a highly liquid market are the Euro - U.S. Dollar (EURUSD), Euro - British Pound (EURGBP) and the U.S Dollar - Japanese Yen (USDJPY). The other three exchange rates serve as a control group to the performance of our forecasting models, because they are cross currencies with less liquid markets. They are the Euro - Hungarian Forint (EURHUF), Euro - Polish Zloty (EURPLN) and the Euro – Romanian Lei (EURRON) crosses.

To construct the daily RV measures we have used high-frequency 5-minute data on the exchange rates downloaded from the Bloomberg (2017) database. Our sample period spans from 2008 to 2017. (The beginning of the period varies across currencies based on the availability of high frequency data in Bloomberg.) Using equation 3 presented in II.1 we constructed the daily realized volatility measures by summing the squared 5-minute log-returns of the exchange rates. Next we have excluded Saturdays and Sundays from our dataset, because on these days there were only few hours where the currencies were traded, and so our RV measures would have been biased downward.
2. Descriptive statistics

In Table 2 we present some of the main statistics of the returns from our sample.

<table>
<thead>
<tr>
<th></th>
<th>EURUSD</th>
<th>EURGBP</th>
<th>USDJPY</th>
<th>EURHUF</th>
<th>EURPLN</th>
<th>EURRON</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A - 5-minute returns</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>0,0000%</td>
<td>0,0000%</td>
<td>0,0000%</td>
<td>0,0000%</td>
<td>0,0000%</td>
<td>0,0000%</td>
</tr>
<tr>
<td>St. Deviation</td>
<td>0,0376%</td>
<td>0,0375%</td>
<td>0,0423%</td>
<td>0,0482%</td>
<td>0,0448%</td>
<td>0,0315%</td>
</tr>
<tr>
<td>Skewness</td>
<td>0,2413</td>
<td>3,4352</td>
<td>-0,9524</td>
<td>-0,4468</td>
<td>0,2266</td>
<td>-0,8786</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>44,1533</td>
<td>435,2152</td>
<td>135,3836</td>
<td>121,7103</td>
<td>837,0804</td>
<td>674,4855</td>
</tr>
<tr>
<td>Observations</td>
<td>681,886</td>
<td>747,703</td>
<td>731,434</td>
<td>742,442</td>
<td>716,453</td>
<td>524,157</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>EURUSD</th>
<th>EURGBP</th>
<th>USDJPY</th>
<th>EURHUF</th>
<th>EURPLN</th>
<th>EURRON</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel B - Daily returns</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>0,0003%</td>
<td>0,0046%</td>
<td>0,0026%</td>
<td>0,0027%</td>
<td>0,0025%</td>
<td>0,0010%</td>
</tr>
<tr>
<td>St. Deviation</td>
<td>0,5992%</td>
<td>0,5816%</td>
<td>0,6461%</td>
<td>0,6301%</td>
<td>0,6167%</td>
<td>0,2791%</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0,0795</td>
<td>0,4852</td>
<td>0,0721</td>
<td>0,8587</td>
<td>0,3954</td>
<td>-0,1260</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>2,2290</td>
<td>7,1734</td>
<td>4,0658</td>
<td>10,9013</td>
<td>7,1087</td>
<td>7,0798</td>
</tr>
<tr>
<td>Observations</td>
<td>2,325</td>
<td>2,553</td>
<td>2,510</td>
<td>2,532</td>
<td>2,552</td>
<td>2,260</td>
</tr>
</tbody>
</table>

Table 2 – The descriptive statistics of 5-minute and daily returns
Own table, based on Bloomberg database (2017)

Panel A of Table 2 present the statistics of 5-minute returns of the whole sample period, while Panel B shows the daily returns’ moments. We can see how the enormous kurtosis of 5-minute returns have become an order of magnitude smaller as we switched to the daily horizon. Nevertheless these values are still significantly different from three which implies that the distributions of returns are non-normal and have fat-tails (with the exception of the EURUSD cross). It is also noteworthy that the EURRON cross is an outlier in our sample in the sense that it has significantly smaller deviation than the rest of the exchange rates.
On Figure 5 we see the long memory property of the EURHUF log-volatility. (Note that the autocorrelation is significant even after 500 lags.) However, of Figure 6 we have plotted the autocorrelation function of the increments of the log-volatility process. It is clear from the figure that the increments of the log-volatility process are negatively correlated, which confirms that the volatility process could be modeled using a fractional Brownian motion, with a Hurst parameter less than 0.5. (In other words the increments of the log-volatility process show an antipersistent
behavior. Note that this is not incompatible with the fact that the log-volatility process itself is a long-memory process.) Furthermore as we have already shown in section II.2 and II.5 the increments of the FX RV measures have similar distributions and multiscaling properties to the RV measures studied in Gatheral et al (2014). From this we assume that the RFSV model should perform well in the case of the chose FX rates as well.

VII. Empirical results

In section IV. and V. we have presented models that we used to predict the RV of the chosen exchange rates: the AR, ARFIMA, HAR-RV and RFSV models. We used a rolling-window technique to forecast the one-, five- and ten-day-ahead realized variances. The size of the rolling window was 250 days, in order to predict next days’ RV using the previous one year’s data. As we predicted the variances several days ahead, we did not evaluate or models based on their ability to forecast the variances five or ten days from given day. Rather we calculated the average variance the models forecasted for that given time period. This way we can give more reliable evaluations of our models, because it’s a much more logical question to ask what the average realized variance will be in the next few days, rather than asking what the volatility will be on a specific day in the future.

To evaluate the performance of the models we have used a measure that is quite common in the literature, called the Q-like measure. Equation 28 shows how this measure is calculated. (Sévi, 2014)

\[
Q_{like} = \frac{RV}{\hat{RV}} - \ln\left(\frac{\hat{RV}}{RV}\right) - 1
\]

(28)

In equation 28 $\hat{RV}_t$ is the forecasted variance measure and $RV$ is the realized measure (either the one-day-ahead or the average $RV$ for $n$-days ahead). In this case we expect better performing models to have a smaller average Q-like value. (Sévi, 2014)
Table 3 – Results of the models
Own Table based on Bloomberg database (2017)

In Table 3 we present the mean Q-values of the forecasting models in the case of every FX cross. We can see that in every case the RFSV model has the smallest Q-like value. This confirms the findings of Gatheral et al (2014), that the RFSV model outperforms all other models.

Table 4 – Results of the models
Own Table based on Bloomberg database (2017)

Looking at the result of the models on longer horizons we get the same results. In every case the RFSV model performs best with the lowest average Q-like measure. We would like to remark two further things about these results. One can see that the performance of the models tend to increase on longer time horizons, which could sound counterintuitive at first. However it is not. In the broad
literature of volatility forecasting this phenomenon is quite ordinary. Many have pointed out that the performance of these models tends to increase as we increase the forecasting horizon up to as much as 50 or 60 days. (Of course after 60 days, the performance of the models starts to decrease.) (Sévi, 2014)

One further remark is that the models seem to perfume worst in the case of the EURRON FX cross. (In every case EURRON has the highest Q-like scores.) This could be because of the way the EURRON crosses volatility evolves. On Figure 7 we can see that the realized variance of the time series is usually very low, with a few days of unusually big volatility. In cases such as this the models perform quite well on most days, and make unusually great errors when the volatility unexpectedly increases. However we conclude that although the models are not so accurate in the case of EURRON, Q-like is still an adequate measure to compare the different models with each other.

In the light of these findings to test the robustness of our results we also ran our models on subsamples of the original time series. These subsamples only contained RV data after January 1st, 2013. This way we exclude from our sample the years where the volatility of exchange rates have been effected by the financial crisis. The results of our subsample analysis are presented in Table 4. It is clear from the table that the RFSV model still proves to be superior compared to the other models. Besides that we would like to point out that neither in Table 3 nor in Table 4 did we find
clear differences between liquid and non-liquid FX rates. In every case the RFSV model proves to be superior.

<table>
<thead>
<tr>
<th>(Subsample) 1-day-ahead forecast</th>
<th>AR</th>
<th>ARFIMA</th>
<th>HAR</th>
<th>RFSV</th>
</tr>
</thead>
<tbody>
<tr>
<td>EURUSD</td>
<td>0,2209</td>
<td>0,2080</td>
<td>0,2047</td>
<td>0,1961</td>
</tr>
<tr>
<td>EURGBP</td>
<td>0,2802</td>
<td>0,2635</td>
<td>0,2595</td>
<td>0,2389</td>
</tr>
<tr>
<td>USDJPY</td>
<td>0,3099</td>
<td>0,2790</td>
<td>0,2799</td>
<td>0,2459</td>
</tr>
<tr>
<td>EURHUF</td>
<td>0,1696</td>
<td>0,1640</td>
<td>0,1626</td>
<td>0,1536</td>
</tr>
<tr>
<td>EURRON</td>
<td>0,4311</td>
<td>0,4366</td>
<td>0,4368</td>
<td>0,3624</td>
</tr>
<tr>
<td>EURPLN</td>
<td>0,1654</td>
<td>0,1657</td>
<td>0,1530</td>
<td>0,1515</td>
</tr>
</tbody>
</table>

Table 5 - Results of the models using RV data after 2013
Own Table, based on Bloomberg database (2017)

Finally to test the significance of our results we ran the Diebold-Mariano (2002) tests to determine whether or not the differences between the models’ performances are statistically significant. The Diebold-Mariano test (D-M test for short) similarly to a t-test compares the forecasting errors of two models and tries to determine whether or not the difference between the two errors is significantly different from zero. In our case we tested the RFSV against every other model using a one-sided D-M test (in the one-sided test our alternative hypothesis was that the errors from the RFSV model are smaller than from the other). We have summed up our results in Table 6.
Table 6 – The result of the one-sided Diebold-Mariano tests
Own Table, based on Bloomberg database (2017)

<table>
<thead>
<tr>
<th></th>
<th>1-day-ahead forecast</th>
<th>5-day-ahead forecast</th>
<th>10-day-ahead forecast</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>AR</td>
<td>ARFIMA</td>
<td>HAR</td>
</tr>
<tr>
<td>EURUSD</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>EURGBP</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>USDJPY</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>EURHUF</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>EURRON</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>EURPLN</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

In Table 6 green cells mark the cases where the RFSV model significantly outperformed a model, and red cells show where the difference was not significant on a 10% significance level. At first it is apparent that RFSV outperformed the AR model in all of the cases, which is not surprising. However as we increased the forecasting horizon, the RFSV was able to significantly outperform less and less models. At the 1-day horizon, it managed to significantly outperform all of the models in three cases (USDJPY, EURRON and EURPLN). However at the 10 day horizon the difference was only significant in the case of EURRON. Considering these results we conclude that although the RFSV model managed to give us somewhat more accurate predictions about future variances, it has to be kept in mind that these results cannot always be considered statistically significant according to the Diebold-Mariano test. This proves to be a serious limitation of this methodology.
From these finding we conclude that although in some cases the RFSV model did not manage to provide statistically significant improvement in forecasting the realized variances of the time series, it still proved to be an excellent predictor of not only equity but also foreign exchange realized variances. And although the in many of the cases our results did not pass the Diebold-Mariano test, the Q-like measures from the models suggest, that RFSV is still far the most accurate predictor of volatility from the four models we chose to examine. And not only that but as a stochastic model of volatility, it can also replicate skew term structures similar to the ones observed in the real world. These properties of the RFSV model in our opinion makes it one of the most advanced stochastic tools as of today, and we think it would be highly rewarding to focus our attention on how we can further improve this model, and on its real life applications for example in risk management, or in pricing.

However there are still a couple of questions left unanswered that give way to further research. Is there a clear economic intuition behind the roughness of FX volatilities? Is there a way to build a microstructure model that can reproduce and explain these findings? In the last section of this paper we are going to shortly discuss some of the main theories from the literature that try to give answers to these questions.

**VIII. Microstructural reasons behind roughness of volatility**

1. **Economic Foundations of the success of the RFSV model**

In this section, we are going to present a market microstructure framework in order to capture the dynamics of trades. The aim of this chapter is to give an economics foundation for the RFSV model, and its superior performance at capturing the dynamics of realized volatility, not to give an extensive review of market microstructure modelling. So, we will focus on ideas and mathematical results that are relevant in understanding the success of the RFSV model.

If one considers a simple pricing model, where prices are built from a sum of independent random events, then the volatility at time $T$ is proportional to the number of event until $T$, to be more precise it is proportional to $\sqrt{N_T}$. Also, in their seminal paper Wyart et al (2007) showed, that there is highly significant empirical relationship between the bid-ask spread and the realized volatility in the market. They have come to the conclusion that most of the volatility comes from
trade impact. These two factors show, that there is a clear connection between modelling the order flow of the market, and the dynamics of the volatility process.

Therefore, we will present standard results from recent order flow modelling, and show how the processes used in these models relate to the main building-block of the RFSV model, fractional Brownian Motion.

2. Market Microstructure of foreign exchange market

In this section, we are going to shortly introduce some microstructural aspects of the foreign exchange market with a special attention to the rise of electronic trading and its effect on the heterogeneity of the trader, and to the phenomenon called the order splitting. This part is heavily based on King et al (2013) and Chen et al (2012).

The authors argue that the FX market is evolving steadily in reaction to the emergence of new electronic trading strategies. The main changes have been the appearance of new types of traders. Traditionally, the FX market has been a wholesale market, where the major players were the corporations that are dealing with international trading, local and regional banks, and central banks. The rise of high-frequency trading brought the appearance of a retail segment that has been growing steadily ever since, representing around 10% of the total trading volume (King et al, 2013). Heimer and Simon (2011) came to the conclusion that these traders are significantly different, than traditional asset managers, because they may base their trading strategies on short-term price forecast, and do not conform the traditional rationality assumptions. All these papers come to the conclusion that the heterogeneity has risen in the FX market in the last two decades, and the FX order flow captures the effect of these heterogeneous beliefs or strategies.

Algorithmic trading has made it possible to optimize a trading strategy with respect to many different dimensions. As mentioned above, the portion of traders that use these technologies in the FX market has risen lately. These institutional investor often optimize their strategy in order to reduce the transaction costs and the price impact of their trades. The phenomenon is often referred to as order splitting. This strategy has risen parallel with the rise of electronic trading.

In the next sections, we show that how this two phenomena shaped the market, and their relevance to the RFSV model.
3. Modelling order flows

In this section our aim is to give a gentle introduction to the idea behind price impact modelling, and highlight the results, that are most relevant to the RFSV model.

The theory of market price formation and the relationship between order-flow of trades and market prices is an important part of current financial research. Many researchers, for example Bouchaud (2009), have shown that the price impact of trades have some universal properties, which corroborates the theory of endogenous price formation. This means, that the impact of trades have autocorrelation, so a significant portion of trades at a given time is a reaction to the impact of previous trades. This contradicts with theory of exogenous price theory, which assumes, that independent exogenous trades drive the price towards an equilibrium level. Following these results, theorists created models, where the price is a result of all trader’s impacts, and some noise, and that’s where the application of the so-called Hawkes process were introduced in market impact modelling.

Hawkes processes were introduced by Hawkes (1971), who tried to model the appearance of earthquakes, and their aftermath. A one-dimensional Hawkes process is a non-homogeneous Poisson process $N(t)$, where the intensity is a stochastic process with a constant base and a self-excitng term.

$$\lambda(t) = \mu + \int_{-\infty}^{t} \varphi(t-s) \, dN(s)$$

(29)

where $\varphi(\tau)$ is called the influence kernel, which describes the effects of past events on the instantaneous intensity rate. This kernel is particularly important, because it represents the effect of previous market orders, to the intensity. A Hawkes process is said to be stable, if $\|\varphi(\tau)\|_1$ is strictly smaller, than one. This condition is similar to stationary condition of a simple first-order auto-regressive process.

The self-exciting nature of these processes, make them a natural choice for order book modelling, because it reproduces the described endogenous price formation. In recent years, it has become a standard to model the order flows of the financial markets with different types of Hawkes processes, for example Bacry and Muzy (2013) modelled equity order book with a four
dimensional Hawkes process, and Rambaldi et al (2015) modelled foreign exchange market activity around macroeconomic news with a simple Hawkes process. The main question that arose after these papers, was how to calibrate Hawkes processes to high-frequency financial data.

V. Filimonov and D. Sornette (2012) recently showed on many examples, that if we calibrate a simple univariate Hawkes process to financial data, we observe, that the $\|\varphi(\tau)\|_1$ is very close to 1. We call these processes nearly unstable Hawkes processes. This empirical phenomenon has very clear connection to the market microstructure of the foreign exchange market, because it coincides with the fact that the rise of high-frequency trading created a high degree of endogeneity. This means that a large portion of trades are endogenous reaction to the trades of other participants. The authors have also introduced a measure of market endogeneity, and also showed that this endogeneity have risen substantially in recent years, which coincides with the rise of high-frequency trading.

Hardiman et al (2013) have also analyzed how Hawkes processes can be fitted to high-frequency financial data. Their main interest was to be able to choose between an exponentially decaying impact kernel, and a power-law kernel. They have come to the conclusion, that in order to capture the fat-tails of returns, the power-law kernel based Hawkes processes should be used. The use of the power-law kernel also introduces long-memory to the order flow. The economic intuition behind this phenomenon is the order-splitting introduced in the previous section.

4. Relevance to the RFSV model
Jaisson and Rosenbaum (2015) have investigated the scaling asymptotic behaviour of nearly unstable Hawkes process with power-law intensity kernels with a tail exponent $\alpha \in (1,2)$. This is relevant, because we have introduced in the previous section that these types of Hawkes processes fit market data best. They chose to analyse such heavy-tailed Hawkes processes, because of the empirical long-memory of the order flow, mainly caused by order-splitting.

They have shown that a properly re-scaled heavy-tailed Hawkes process converges to an integrated fractional Brownian Diffusion (to be more precise an integrated fractional Cox-Ingersoll-Ross process). Also, they have shown that the Hurst-parameter of the limiting process is smaller than 0.5, so the paths of this process are very irregular. Combining this result with the previous sections
of this chapter shows that the volatility should be very irregular, which coincides with the results about the empirical smoothness of the realized volatility that we have calculated in this work.

In short, we have shown in this chapter that the empirical roughness of the realized volatility, hence the superior performance of the RFSV model can be explained by two factors. First, the high and raising endogeneity of the price formation in the financial market, which can be explained by the widespread use of high-frequency trading. Second, the long-memory of the order flow process, which can be explained by the existence of order-splitting.

**IX. Conclusion**

In this study, we have analyzed the realized volatility processes of six different currencies. We have had two main research questions in this paper. First, we wanted to see, whether the volatility of currency markets exhibit the same irregularities as the volatility of the equity markets. Second, we wanted to see, whether the RFSV model built by Gatheral et al (2014) performs for currencies as well as for equity data. To answer these questions, first we have introduce some stylized facts of the volatility of financial time series. Then, we have introduced the concept of implied volatility, and the inability of standard financial models to capture the dynamics of the implied volatility surface. We have emphasized that, there are major differences between the implied volatility surface of FX and equity data. Here we have also referred to Fukasawa (2011), who has shown that models based on fractional Brownian motion are able to reproduce these dynamics. This result gave financial relevance to the main model used in this paper. Then, we have introduced the RFSV model, and other econometric models, that were used as a benchmark, when we analyzed the ability of RFSV model to forecast the realized volatility. Finally we used these models on our dataset, which was composed of high-frequency data from liquid and illiquid currencies (these were the EURUSD, USDJPY, EURGBP, EURPLN, EURHUF and EURRON).

We have found encouraging answers for our two research questions. The empirical analyses of the realized volatility processes revealed that, the smoothness parameter of the volatilities are generally quite low, definitely lower, than 0.5. This means that in the RFSV model the Hurst parameter of the Fractional Brownian motion is less, than 0.5, which means that, the paths are considered rougher, than the paths of the Brownian motion. This coincides with the results of Gatheral et al (2014). Also, the results of the forecasting showed that, the RFSV model outperforms
the benchmark models both on the whole dataset, and on a relevant subsample. Our findings show that the RFSV model serves as an outstanding forecasting model of not only equity volatilities but also FX volatilities. Besides its ability to make accurate predictions it has many other appealing features that make it a state of the art volatility model with many possible applications in the world of financial modelling, pricing and risk management.
X. References


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